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A MORE FUNDAMENTAL APPROACH TO DAMAGED ELASTIC STRESS–STRAIN RELATIONS

Q.-C. HE and A. CURNIER
Laboratoire de Mécanique Appliquée, Département de Mécanique,
Ecole Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland

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Transmitted by Professor H. D. Bui

Abstract—For many engineering problems it can be assumed that the damaged material elastic response at a fixed damage state is linear and hyperelastic. With this assumption, a systematic and rigorous approach for formulating damaged elastic stress–strain relations is presented. The approach is based on the Fourier series representations of two naturally defined damage orientation distribution functions and on a theorem on elasticity tensors, and developed resorting neither to the notion of effective stress (strain) nor to the hypothesis of strain (stress) equivalence nor to that of elastic energy equivalence. The proposed approach is finally illustrated by applying it to initially isotropic materials and to unidirectional fiber-reinforced composites.

1. INTRODUCTION

In continuum damage mechanics (see e.g. Chaboche, 1988; Krajcinovic, 1989), the most widely used constitutive models are probably the ones based on the thermodynamics of irreversible processes with internal variables (see e.g. Gurtin, 1972; Germain et al., 1983). Any damage model of this kind is essentially composed of three parts:

(i) choice of the damage (internal) variables macroscopically characterizing material microstructural damage states;
(ii) formulation of the damaged material behavior at a given damage state;
(iii) establishment of the evolution equations for the damage variables, relative to a loading history.

The present work aims at investigating parts (i) and (ii), with particular reference to the mathematical nature of damage variables and to the damaged elastic response.

As has been pointed out by several authors (see e.g. Krajcinovic, 1989; Ju, 1990), the predictive utility of a continuum damage model with internal variables depends largely on the degree of approximation with which the geometric characters and macroscopic effects of the microdefects (microvoids and microcracks) are described by the chosen damage variables. One of the important problems to be resolved in the present development of continuum damage mechanics is the lack of uniformity and rigor in the choice of damage variables (see Rabier, 1989, for a critical review of this matter). In the present work, trying to find a mathematical solution to this problem, we are led to define damage orientation distribution functions and to study the radially symmetric scalar-valued functions. The mathematical nature of damage variables is then brought out by establishing a representation lemma for the radially symmetric scalar-valued functions, applying it to damage orientation distribution functions and then expanding them into Fourier series.

In most of the existing damage theories, the damaged elastic strain–stress (or stress–strain) response is formulated by using the notion of *effective stress* (*strain*) and the hypothesis of *strain* (*stress*) *equivalence* or *stress-energy* (*strain-energy*) *equivalence* (Lemaitre and Chaboche, 1985; Cordebois and Sidoroff, 1979). In the present work, a different and more fundamental approach initiated by Ladevèze (1983) is substantially developed. This approach rests on a theorem on (undamaged or damaged) elasticity tensors and requires neither the above notions nor the relevant hypotheses. It is applicable to both initially isotropic and anisotropic elastic materials and has the advantage of neglecting only the terms related to higher spherical harmonics. To be more specific, a general procedure for obtaining a finite description of any material damage state is presented in coordinate-free form; material symmetries are systematically taken into account; the one-to-one relationship between a pair of modulus orientation distribution functions and a fourth-order elasticity tensor is clearly stated (Theorem 2) and provided with an existence proof; a general expression for the damaged elasticity in terms of damage variables and undamaged (isotropic or anisotropic) elasticity tensors is specified [eqn (49) together with eqns (45)–(48)]; a detailed discussion of restrictions on the choice of damage variables is given; a constructive and important example, the damage of unidirectional fibre-reinforced composites, is treated. To a certain extent, these new elements complete the approach introduced by Ladevèze (1983, 1990, 1993) and make it more explicit and more flexible.

An outline of the paper is as follows. In Section 2 we give a brief account of the adopted notation. Section 3 contains a representation lemma for radially symmetric scalar-valued functions and a general description of how to expand a square-integrable scalar-valued orientation distribution function in a convergent Fourier series. In Section 4, we first define the elongation and bulk modulus orientation distribution functions and then, with the help of the irreducible decomposition of elasticity tensors, show that there exists a one-to-one correspondence between these two functions and an elasticity tensor. Section 5 is devoted to the development of a systematic approach to damaged elastic stress–strain relations. After making the hypothesis that the damaged elastic response at a fixed damage state is linear and hyperelastic, two damage orientation distribution functions are introduced and expanded in two Fourier series, the coefficients of which behave as natural damage variables. Based on the main results of Section 4, a general and consistent method of constructing the damaged elasticity tensor is described. The developed approach is then illustrated by applying it to initially isotropic materials in Section 6 and to unidirectional fibre-reinforced composites in Section 7. Some concluding remarks are given in Section 8.

2. NOTATION

Direct or coordinate-free notation will be employed as much as possible. This permits us to write formulae in rather clear and compact forms.

As a general rule, light-face (Greek or Latin) letters denote scalars; bold-face minuscules and majuscules designate vectors and second-order tensors, respectively; outline letters are reserved for fourth-order tensors; script majuscules stand for sets, spaces, domains or groups. The principal notations are now introduced.

Let $\mathscr{V}$ be a three-dimensional inner-product space over the reals $\mathscr{R}$. We denote by $\mathscr{L}$ the space of all linear transformations (second-order tensors) on $\mathscr{V}$ and by $\mathbb{L}$ the space of all linear transformations (fourth-order tensors) on $\mathscr{L}$. The inner products of $\mathscr{V}$, $\mathscr{L}$ and $\mathbb{L}$ are labelled as follows: $\mathbf{a}.\mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in \mathscr{V}$, $\mathbf{A}:\mathbf{B}$ for $\mathbf{A}, \mathbf{B} \in \mathscr{L}$, and $\mathbb{A} :: \mathbb{B}$ for $\mathbb{A}, \mathbb{B} \in \mathbb{L}$.

The transposes $\mathbf{A}^{\mathrm{T}}$ and $\mathbb{A}^{\mathrm{T}}$ of $\mathbf{A} \in \mathscr{L}$ and $\mathbb{A} \in \mathbb{L}$ are defined by

$$\mathbf{A}^{\mathrm{T}}\mathbf{x}.\mathbf{y} := \mathbf{x}.\mathbf{A}\mathbf{y}, \forall \mathbf{x}, \mathbf{y} \in \mathscr{V}; \quad \mathbb{A}^{\mathrm{T}}\mathbf{X}:\mathbf{Y} := \mathbf{X}:\mathbb{A}\mathbf{Y}, \forall \mathbf{X}, \mathbf{Y} \in \mathscr{L}.$$

We say that $\mathbf{A}$ is *symmetric* if $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$, *positive (semi-) definite* if $\mathbf{x}.\mathbf{A}\mathbf{x} > 0$ ($\geqslant 0$) for $\mathbf{x} \neq 0$. Similarly, $\mathbb{A}$ is said to have the *major symmetry* if $\mathbb{A}^{\mathrm{T}} = \mathbb{A}$ and to be *positive (semi-) definite* if $\mathbf{X}:\mathbb{A}\mathbf{X} > 0$ ($\geqslant 0$) for $\mathbf{X} \neq 0$.

Given $\mathbf{a}, \mathbf{b} \in \mathscr{V}$, $\mathbf{A}, \mathbf{B} \in \mathscr{L}$ and $\mathbb{C} \in \mathbb{L}$, we define $\mathbf{a} \otimes \mathbf{b}$, $\mathbf{A} \otimes \mathbf{B}$, $\mathbf{A} \underline{\otimes} \mathbf{B}$, $\mathbf{A} \bar{\otimes} \mathbf{B}$, $\mathbb{C} \otimes \mathbf{A}$ and $\mathbf{A} \otimes \mathbb{C}$ through

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{x} := (\mathbf{b}.\mathbf{x})\mathbf{a}, \forall \mathbf{x} \in \mathscr{V}; \quad (\mathbf{A} \otimes \mathbf{B})\mathbf{X} := (\mathbf{B}:\mathbf{X})\mathbf{A}, \forall \mathbf{X} \in \mathscr{L};$$

$$(\mathbf{A} \underline{\otimes} \mathbf{B})(\mathbf{x} \otimes \mathbf{y}) := (\mathbf{A}\mathbf{x}) \otimes (\mathbf{B}\mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in \mathscr{V};$$

$$(\mathbf{A} \bar{\otimes} \mathbf{B})(\mathbf{x} \otimes \mathbf{y}) := (\mathbf{A}\mathbf{y}) \otimes (\mathbf{B}\mathbf{x}), \forall \mathbf{x}, \mathbf{y} \in \mathscr{V};$$

$$(\mathbb{C} \otimes \mathbf{A})\mathbf{X} := (\mathbf{A}:\mathbf{X})\mathbb{C}, \forall \mathbf{X} \in \mathscr{L}; \quad (\mathbf{A} \otimes \mathbb{C})\mathbb{X} := (\mathbb{C} :: \mathbb{X})\mathbf{A}, \forall \mathbb{X} \in \mathbb{L}.$$

Let $\mathscr{V} \otimes \mathscr{V}$ designate the set of all finite linear combinations of $\mathbf{a} \otimes \mathbf{b}$ with $\mathbf{a}, \mathbf{b} \in \mathscr{V}$. Then $\mathscr{L}$ can be identified with $\mathscr{V} \otimes \mathscr{V}$ via $(\mathbf{a} \otimes \mathbf{b})\mathbf{x} := (\mathbf{b}.\mathbf{x})\mathbf{a}$ for all $\mathbf{x} \in \mathscr{V}$. Analogously, $\mathbb{L}$ can be identified with $\mathscr{L} \otimes \mathscr{L}$. The products $\mathbf{A} \underline{\otimes} \mathbf{B}$ and $\mathbf{A} \bar{\otimes} \mathbf{B}$ are of Kronecker type. With the help of the identities $(\mathbf{Ax}) \otimes (\mathbf{By}) \equiv \mathbf{A}(\mathbf{x} \otimes \mathbf{y})\mathbf{B}^{\mathrm{T}}$ and $(\mathbf{x} \otimes \mathbf{y})^{\mathrm{T}} \equiv \mathbf{y} \otimes \mathbf{x}$, it can be verified that

$$(\mathbf{A} \underline{\otimes} \mathbf{B})\mathbf{X} = \mathbf{AXB}^{\mathrm{T}}, \forall \mathbf{X} \in \mathscr{L}; \quad (\mathbf{A} \bar{\otimes} \mathbf{B})\mathbf{X} = \mathbf{AX}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}, \forall \mathbf{X} \in \mathscr{L}.$$

In particular, the transposition mapping $\mathbb{T}$ and the *symmetric* fourth-order tensor identity $\mathbb{I}$ are given by

$$\mathbb{T} = \mathbf{I} \bar{\otimes} \mathbf{I}, \quad \mathbb{I} = \tfrac{1}{2}(\mathbf{I} \underline{\otimes} \mathbf{I} + \mathbf{I} \bar{\otimes} \mathbf{I}),$$

where $\mathbf{I}$ presents the second-order identity tensor.

In what follows, $\mathbf{n}$ will always denote a three-dimensional *unit* vector, $\mathscr{S}$ the *unit sphere* $\{\mathbf{n} \in \mathscr{V} \mid \|\mathbf{n}\| = 1\}$ and $\mathbf{N}$ an element of $\mathscr{N} := \{\mathbf{n} \otimes \mathbf{n} | \mathbf{n} \in \mathscr{S}\}$. We also use the notations:

$$(\mathbf{A} \otimes \mathbb{C}) ::: (\mathbf{B} \otimes \mathbb{D}) := (\mathbf{A} : \mathbf{B})(\mathbb{C} :: \mathbb{D}); \quad (\mathbb{C} \otimes \mathbf{A}) ::: (\mathbb{D} \otimes \mathbf{B}) := (\mathbb{C} :: \mathbb{D})(\mathbf{A} : \mathbf{B});$$

$$(\mathbb{A} \otimes \mathbb{B}) :::: (\mathbb{C} \otimes \mathbb{D}) := (\mathbb{A} :: \mathbb{C})(\mathbb{B} :: \mathbb{D}).$$

3. SCALAR-VALUED ORIENTATION DISTRIBUTION FUNCTIONS

3.1. *Representation of the radially symmetric functions*

A function $\rho : \mathscr{V} \to \mathscr{R}$ is said to be *radially symmetric* if

$$\rho(\mathbf{v}) = \rho(-\mathbf{v}), \quad \forall \mathbf{v} \in \mathscr{V}. \tag{1}$$

As will be seen, the functions possessing such a property play an important role in our work. We now give a preliminary result about them.

Lemma 1. A function ρ from $\mathscr{V}$ to $\mathscr{R}$ is radially symmetric if and only if there exists a function $\hat{\rho} : \mathscr{V} \otimes \mathscr{V} \to \mathscr{R}$ such that

$$\rho(\mathbf{v}) = \hat{\rho}(\mathbf{v} \otimes \mathbf{v}), \quad \forall \mathbf{v} \in \mathscr{V}. \qquad \bullet$$

Proof. Sufficiency is trivial and we need only show necessity. Given any $\mathbf{v} \in \mathscr{V}$, it is always possible to choose an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ such that $\mathbf{v} = v_1\mathbf{e}_1$. Suppose eqn (1) holds and define $\mathbf{V} := \mathbf{v} \otimes \mathbf{v}$. Noting that $\mathbf{V}$ is positive semi-definite, the square-root $\sqrt{\mathbf{V}}$ of $\mathbf{V}$ is then known to be well-defined, unique and equal to $|v_1|(\mathbf{e}_1 \otimes \mathbf{e}_1)$. Thus, we can write

$$\begin{aligned}\rho(\mathbf{v}) &= \rho(v_1\mathbf{e}_1) = \rho(-v_1\mathbf{e}_1) = \rho(|v_1|\mathbf{e}_1)\\ &= \rho[|v_1|(\mathbf{e}_1 \otimes \mathbf{e}_1)\mathbf{e}_1] = \rho(\sqrt{\mathbf{V}}\mathbf{e}_1) = \rho[\sqrt{\mathbf{V}}\hat{\mathbf{e}}_1(\mathbf{V})],\end{aligned}$$

where $\mathbf{e}_1 = \hat{\mathbf{e}}_1(\mathbf{V})$, the function $\hat{\mathbf{e}}_1(\mathbf{V})$ being defined through $\mathbf{V}\mathbf{e}_1 = \|\mathbf{V}\|\mathbf{e}_1$. Therefore, there exists a function $\hat{\rho}(\mathbf{v} \otimes \mathbf{v}) = \hat{\rho}(\mathbf{V}) := \rho(\sqrt{\mathbf{V}}\hat{\mathbf{e}}_1(\mathbf{V}))$, such that $\rho(\mathbf{v}) = \hat{\rho}(\mathbf{v} \otimes \mathbf{v})$ for all $\mathbf{v} \in \mathscr{V}$. $\bullet$

A radially symmetric function as precedently defined is in fact *isotropic* relative to the group $\mathscr{G} := \{\mathbf{I}, -\mathbf{I}\}$ and *anisotropic* relative to the group $\mathscr{O}(3)$ of all three-dimensional orthogonal tensors. Lemma 1 can easily be shown to hold in the case where $\mathscr{V}$ is any finite dimensional vector space. In particular, setting $\mathscr{V} = \mathscr{R}$, we obtain the classical representation result for the *even* functions from $\mathscr{R}$ to $\mathscr{R}$.

3.2. *Expansion of the scalar-valued orientation distribution functions*

Let φ be some macroscopic scalar property of a material. At a given instant, φ generally depends on the material point, identified with the reference position vector $\mathbf{x}$, and on the orientation, specified by the unit vector $\mathbf{n}$; that is, $\varphi = \varphi(\mathbf{x}, \mathbf{n})$. Since only the dependence of φ on $\mathbf{n}$ is concerned in subsequent investigations, it is convenient to consider $\mathbf{x}$ as fixed and drop the dependence of φ on $\mathbf{x}$. Then we write

$$\varphi = f(\mathbf{n}), \quad f: \mathscr{S} \to \mathscr{R} \tag{2}$$

and call f, the scalar-valued function defined on the unit sphere $\mathscr{S}$, the *orientation distribution function* (o.d.f.) of the property φ. Concretely φ may be the effective surface density of microdefects, Young's modulus, the wave speed, the electrical resistivity, the fatigue limit, etc. (Lemaitre and Dufailly, 1987).

The function $f(\mathbf{n})$ must satisfy the condition

$$f(\mathbf{n}) = f(-\mathbf{n}), \quad \forall \mathbf{n} \in \mathscr{S}, \tag{3}$$

because any material property φ in a direction is independent of the geometrical choice made between $\mathbf{n}$ and $-\mathbf{n}$ for defining that direction. According to lemma 1, the invariance requirement (3) is satisfied if and only if there exists a function $\hat{f}$ from $\mathscr{N}$ to $\mathscr{R}$ such that

$$f(\mathbf{n}) = \hat{f}(\mathbf{n} \otimes \mathbf{n}) = \hat{f}(\mathbf{N}), \quad \forall \mathbf{n} \in \mathscr{S}. \tag{4}$$

From now on, we need only study the function $\hat{f}(\mathbf{N})$ for condition (3) to be automatically verified.

Assume $\hat{f}(\mathbf{N})$ to be *square-integrable*:

$$\int_{\mathscr{S}} |\hat{f}(\mathbf{N})|^2 \, \mathrm{d}a < +\infty, \tag{5}$$

where $\mathrm{d}a$ ($= \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$) is an infinitesimal surface element of the unit sphere $\mathscr{S}$. It is then known (Vilenkin, 1969; Bunge, 1982; Jones, 1985) that $\hat{f}(\mathbf{N})$ can be expanded in the following Fourier series:

$$\begin{aligned} \hat{f}(\mathbf{N}) &= f_0(\mathbf{N}) + f_1(\mathbf{N}) + f_2(\mathbf{N}) + \cdots \\ &= g + \mathbf{G}' : \mathbf{F}(\mathbf{N}) + \mathbb{G}' :: \mathbb{F}(\mathbf{N}) + \cdots, \quad \forall \mathbf{N} \in \mathscr{N}, \end{aligned} \tag{6}$$

which is *convergent in mean*, i.e.

$$\lim_{n \to \infty} \int_{\mathscr{S}} |\hat{f}(\mathbf{N}) - s_n(\mathbf{N})|^2 \, \mathrm{d}a = 0, \quad s_n(\mathbf{N}) := f_0(\mathbf{N}) + f_1(\mathbf{N}) + \cdots + f_n(\mathbf{N}). \tag{7}$$

In eqn (6), $\{1, \mathbf{F}(\mathbf{N}), \mathbb{F}(\mathbf{N}), \ldots\}$ are *generalized spherical harmonics* (Kanatani, 1984; Onat, 1984; Jones, 1985) and form a *complete orthogonal basis* for the square-integrable functions on $\mathscr{S}$.

For the present work, the first two tensor spherical harmonics $\mathbf{F}(\mathbf{N})$ and $\mathbb{F}(\mathbf{N})$ are of particular interest, which, in view of the two tensor products of Kronecker-type introduced in Section 2, can be written in the *coordinate-free* forms:

$$\mathbf{F}(\mathbf{N}) = \mathbf{N} - \tfrac{1}{3}\mathbf{I}; \tag{8a}$$

$$\mathbb{F}(\mathbf{N}) = \mathbf{N} \otimes \mathbf{N} - \tfrac{1}{7}(\mathbf{I} \otimes \mathbf{N} + \mathbf{N} \otimes \mathbf{I} + \mathbf{I} \underline{\otimes} \mathbf{N} + \mathbf{N} \underline{\otimes} \mathbf{I} + \mathbf{I} \overline{\otimes} \mathbf{N} + \mathbf{N} \overline{\otimes} \mathbf{I})$$

$$+\tfrac{1}{35}(\mathbf{I}\otimes\mathbf{I}+\mathbf{I}\,\underline{\otimes}\,\mathbf{I}+\mathbf{I}\,\bar{\otimes}\,\mathbf{I}). \tag{8b}$$

The orthogonality of the basis functions $\{1, \mathbf{F}(\mathbf{N}), \mathbb{F}(\mathbf{N}), \ldots\}$ means that

$$\int_{\mathscr{S}} \mathbf{F}(\mathbf{N})\,\mathrm{d}a = \mathbf{0}, \quad \int_{\mathscr{S}} \mathbb{F}(\mathbf{N})\,\mathrm{d}a = \mathbb{O}$$

$$\int_{\mathscr{S}} \mathbf{F}(\mathbf{N})\otimes\mathbb{F}(\mathbf{N})\,\mathrm{d}a = \int \mathbb{F}(\mathbf{N})\otimes\mathbf{F}(\mathbf{N})\,\mathrm{d}a = \mathbb{O}_6, \ldots \tag{9}$$

where $\mathbb{O}_6$ denotes the sixth-order zero tensor. It is important to remark that $\mathbf{F}(\mathbf{N})$ is *symmetric and traceless* (s.t.):

$$\mathbf{F}^{\mathrm{T}} = \mathbf{F}; \quad \mathbf{I}:\mathbf{F} = 0, \tag{10a}$$

and that $\mathbb{F}(\mathbf{N})$ is *completely symmetric and traceless* (c.s.t.):

$$\mathbb{F} = (\mathbf{I}\,\bar{\otimes}\,\mathbf{I})\mathbb{F} = \mathbb{F}^{\mathrm{T}}; \quad (\mathbf{Y}\,\bar{\otimes}\,\mathbf{X})::\mathbb{F} = (\mathbf{Y}\,\bar{\otimes}\,\mathbf{X}^{\mathrm{T}})::\mathbb{F}, \quad \forall\mathbf{X},\mathbf{Y}\in\mathscr{L}; \quad \mathbb{F}\mathbf{I} = \mathbf{0}. \tag{10b}$$

(In component forms, these conditions are written as $\mathbb{F}_{ijkl} = \mathbb{F}_{jikl} = \mathbb{F}_{klij} = \mathbb{F}_{ikjl}$ and $\mathbb{F}_{ijkk} = \mathbf{0}_{ij}$.) The first three expansion coefficients of eqn (6) can be determined from $\hat{f}(\mathbf{N})$ via the integrals (see e.g. Kanatani, 1984):

$$g = \frac{1}{4\pi}\int_{\mathscr{S}} \hat{f}(\mathbf{N})\,\mathrm{d}a, \quad \mathbf{G}' = \frac{15}{8\pi}\int_{\mathscr{S}} \hat{f}(\mathbf{N})\,\mathbf{F}(\mathbf{N})\,\mathrm{d}a$$

$$\mathbb{G}' = \frac{315}{32\pi}\int_{\mathscr{S}} \hat{f}(\mathbf{N})\,\mathbb{F}(\mathbf{N})\,\mathrm{d}a. \tag{11}$$

Due to eqn (10), $\mathbf{G}'$ turns out to be s.t. and $\mathbb{G}'$ to be c.s.t. With these properties, in the most general case, $\mathbf{G}'$ and $\mathbb{G}'$ contain *five* and *nine* independent components, respectively.

It is readily seen from eqns (6) and (7) that any square-integrable o.d.f. $\hat{f}(\mathbf{N})$ is fully characterized by its scalar and tensor expansion coefficients $\{g, \mathbf{G}', \mathbb{G}', \ldots\}$. If only the leading terms (for example, the first three ones) of the series expansion, eqn (6), are retained, a *finite* or *discrete* description is then obtained for $\hat{f}(\mathbf{N})$. Theoretically speaking, the accuracy of such a description increases with the number of the leading terms being employed; in practice, the maximum value of this number is determined by the degree of accuracy with which the directional data of the property φ are experimentally acquired. This problem will be further discussed in subsequent investigations.

The importance of Lemma 1 resides in the fact that, when combined with Fourier series expansions, it immediately reveals that *only* the tensors of *zero or even* orders are usable for a finite description of the o.d.f. of a scalar-valued physical or mechanical property φ. The same result was deduced by Onat (Onat, 1984; Onat and Leckie, 1988) in a different way.

In addition to the general symmetry requirement (3), $f(\mathbf{n})$ and then $\hat{f}(\mathbf{N})$ must also fulfil the material symmetry conditions. This has the consequence that the number of the independent components of the tensor expansion coefficients $\{\mathbf{G}', \mathbb{G}', \ldots\}$ is reduced. For the moment, we do not examine this problem in depth and only consider an example to have an idea. Suppose the material in question is isotropic. Then, $f(\mathbf{n}) = f(\mathbf{Qn}) = \hat{f}(\mathbf{QNQ}^{\mathrm{T}})$ for each $\mathbf{n}\in\mathscr{S}$ and all $\mathbf{Q}\in\mathscr{O}(3)$. If limited to studying the consequence of this isotropy condition on $\mathbf{G}'$, we need only write $\mathbf{QG}'\mathbf{Q}^{\mathrm{T}} = \mathbf{G}'$ for all $\mathbf{Q}\in\mathscr{O}(3)$. It is well known that every isotropic second-order tensor is equal to the identity tensor $\mathbf{I}$ multiplied by a scalar. This result together with the traceless of $\mathbf{G}'$ allows us to conclude that $\mathbf{G}' = \mathbf{0}$.

4. A THEOREM ON ELASTICITY TENSORS

In the previous section we studied the orientational dependence of a scalar-valued material property φ via its Fourier series representation. In this section, attention being devoted to linear elasticity, we first introduce two o.d.f.s for elastic properties and then show that there exists an one-to-one correspondence between these functions and an elasticity tensor.

4.1. *Elongation and bulk modulus* o.d.f.s

Denoting by $\mathbf{S}$ the (Cauchy) stress tensor and by $\mathbf{E}$ the (infinitesimal) strain tensor, the stress–strain relation for a *linear hyperelastic* material takes the simple form

$$\mathbf{S} = \mathbb{K}\mathbf{E}, \tag{12}$$

where $\mathbb{K}$, called the *elasticity tensor* and assumed to be *positive definite*, possesses two *minor* symmetries resulting from those of $\mathbf{E}$ and $\mathbf{S}$, and a *major* one due to the hypothesis of hyperelasticity:

$$\mathbb{K} = (\mathbf{I} \bar{\otimes} \mathbf{I})\mathbb{K} = \mathbb{K}(\mathbf{I} \bar{\otimes} \mathbf{I})\,; \quad \mathbb{K} = \mathbb{K}^{\mathrm{T}}. \tag{13}$$

It is well known that, conditioned by eqn (13), $\mathbb{K}$ includes 21 independent constants in the general case.

Consider a *pure* (*or uniaxial*) *elongation* test in the direction $\mathbf{n}$ of the material. In such a case, the strain tensor $\mathbf{E}$ has the form

$$\mathbf{E} = \mathbf{E}(\mathbf{N}) = \varepsilon\mathbf{N}, \quad (\mathbf{N} = \mathbf{n} \otimes \mathbf{n}), \tag{14}$$

where $\mathbf{E}$ is of rank one and ε is the only non-zero principal strain. The stress response is given by eqn (12):

$$\mathbf{S} = \mathbf{S}(\mathbf{N}) = \mathbb{K}\mathbf{E}(\mathbf{N}) = \varepsilon\mathbb{K}\mathbf{N}. \tag{15}$$

The corresponding *normal stress* $\sigma(\mathbf{N}) = \mathbf{N}:\mathbf{S}(\mathbf{N})$ in the direction $\mathbf{n}$ and the relevant *bulk stress* ($p(\mathbf{N}) = \mathbf{I}:\mathbf{S}(\mathbf{N})$) are then deduced to be

$$\sigma(\mathbf{N}) = \varepsilon\mathbf{N}:\mathbb{K}\mathbf{N}, \quad p(\mathbf{N}) = \varepsilon\mathbf{I}:\mathbb{K}\mathbf{N}. \tag{16}$$

Inspired by a work of Ladevèze (1983), we define the *elongation and bulk modulus* o.d.f.s $K(\mathbf{N})$ and $\kappa(\mathbf{N})$ as

$$K(\mathbf{N}) := \frac{\mathbf{N}:\mathbf{S}(\mathbf{N})}{\mathbf{N}:\mathbf{E}(\mathbf{N})} = \frac{\sigma(\mathbf{N})}{\varepsilon}, \quad \kappa(\mathbf{N}) := \frac{\operatorname{tr}(\mathbf{S}(\mathbf{N}))}{\operatorname{tr}(\mathbf{E}(\mathbf{N}))} = \frac{p(\mathbf{N})}{\varepsilon}. \tag{17}$$

Substituting eqn (16) into eqn (17) yields

$$K(\mathbf{N}) = \mathbf{N}:\mathbb{K}\mathbf{N}, \quad \kappa(\mathbf{N}) = \mathbf{I}:\mathbb{K}\mathbf{N}. \tag{18}$$

In particular, if the elastic material in question is *isotropic*, the functions $K(\mathbf{N})$ and $\kappa(\mathbf{N})$ become constant:

$$K(\mathbf{N}) = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}, \quad \kappa(\mathbf{N}) = \frac{E}{1-2\nu}, \tag{19}$$

where E is Young's modulus and ν Poisson's ratio. With eqn (19) and the positive definiteness of $\mathbb{K}$ (which is equivalent to $E > 0$ and $-1 < \nu < 0.5$), the geometrical pictures of $K(\mathbf{N})$ and $\kappa(\mathbf{N})$ are two spheres of *different* radii (except for $\nu = 0$).

4.2. *The irreducible decomposition of elasticity tensors*

When the considered material is not isotropic, the expressions of $K(\mathbf{N})$ and $\kappa(\mathbf{N})$ given by eqn (18) are much more complicated than those of eqn (19). As will be seen, for bringing out the essential properties of $K(\mathbf{N})$ and $\kappa(\mathbf{N})$ in anisotropic cases, it is efficient to carry out the *irreducible decomposition* of $\mathbb{K}$.

Let us first note that all the fourth-order tensors $\mathbb{K}$ satisfying eqn (13) form a vector space of dimension 21, denoted by $\mathscr{T}$. The sets $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_3$, $\mathscr{T}_4$ and $\mathscr{T}_5$, defined through

$$\begin{aligned}
&\mathscr{T}_1 := \{\mathbb{T} = a\mathbf{I}\otimes\mathbf{I} \mid a\in\mathscr{R}\}, \quad \mathscr{T}_2 := \{\mathbb{T} = b(\mathbf{I}\,\underline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{I}) \mid b\in\mathscr{R}\},\\
&\mathscr{T}_3 := \{\mathbb{T} = \mathbf{I}\otimes\mathbf{X}'+\mathbf{X}'\otimes\mathbf{I} \mid \mathbf{X}'\in\mathscr{S}, \mathbf{I}:\mathbf{X}' = 0\},\\
&\mathscr{T}_4 := \{\mathbb{T} = \mathbf{I}\,\underline{\otimes}\,\mathbf{X}'+\mathbf{X}'\,\underline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{X}'+\mathbf{X}'\,\overline{\otimes}\,\mathbf{I} \mid \mathbf{X}'\in\mathscr{S}, \mathbf{I}:\mathbf{X}' = 0\},\\
&\mathscr{T}_5 := \{\mathbb{T}\in\mathscr{T} \mid \mathbb{T} \text{ is c.s.t.}\},
\end{aligned} \tag{20}$$

are subspaces of $\mathscr{T}$, with $\dim(\mathscr{T}_1) = \dim(\mathscr{T}_2) = 1$, $\dim(\mathscr{T}_3) = \dim(\mathscr{T}_4) = 5$ and $\dim(\mathscr{T}_5) = 9$. It can be shown (see e.g. Backus, 1970; Spencer, 1970; Pratz, 1983; Jones, 1985) that the space $\mathscr{T}$ is the *direct sum* of $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_3$, $\mathscr{T}_4$ and $\mathscr{T}_5$:

$$\mathscr{T} = \mathscr{T}_1 \oplus \mathscr{T}_2 \oplus \mathscr{T}_3 \oplus \mathscr{T}_4 \oplus \mathscr{T}_5; \tag{21}$$

that is, any $\mathbb{K}\in\mathscr{T}$ admits the following decomposition

$$\begin{aligned}
\mathbb{K} = \alpha\mathbf{I}\otimes\mathbf{I}+\beta(\mathbf{I}\,\underline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{I})+\mathbf{I}\otimes\mathbf{A}'+\mathbf{A}'\otimes\mathbf{I}&\\
+\mathbf{I}\,\underline{\otimes}\,\mathbf{B}'+\mathbf{B}'\,\underline{\otimes}\,\mathbf{I}+\mathbf{I}\,\overline{\otimes}\,\mathbf{B}'+\mathbf{B}'\,\overline{\otimes}\,\mathbf{I}+\mathbb{K}'.&
\end{aligned} \tag{22}$$

Here α and β are two scalars, $\mathbf{A}'$ and $\mathbf{B}'$ are two s.t. second-order tensors, and $\mathbb{K}'$ is a c.s.t. fourth-order tensor containing, in the general case, *nine* independent components. How to calculate α, β, $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbb{K}'$ starting from $\mathbb{K}$ can be found in the paper by Cowin (1989) and is, in Appendix 1, detailed when $\mathbb{K}$ is transversely isotropic.

The decomposition (21) is *irreducible* in the following sense. Recall that, by definition (see e.g. Vilenkin, 1969), a (linear) representation $\mathbf{R}(\mathscr{G})$ of a group $\mathscr{G}$ in an n-dimensional vector space $\mathscr{V}$ is a *homomorphism* $\mathbf{R}$ from $\mathscr{G}$ to the set $\mathscr{L}^+(\mathscr{V})$ of all invertible linear transformations on $\mathscr{V}$. In other words, each element g of $\mathscr{G}$ is associated with an element $\mathbf{R}(g)$ of $\mathscr{L}^+(\mathscr{V})$ such that $\mathbf{R}(gh) = \mathbf{R}(g)\mathbf{R}(h)$ for $g, h\in\mathscr{G}$. A subspace $\mathscr{W}\subseteq\mathscr{V}$ is said to be *invariant* under $\mathscr{G}$, if $\mathbf{R}(g)\mathbf{w}\in\mathscr{W}$ for every $\mathbf{w}\in\mathscr{W}$ and for each $g\in\mathscr{G}$. A representation in $\mathscr{V}$ is called *irreducible* whenever the only invariant subspaces of $\mathscr{V}$ are the trivial ones $\{\mathbf{0}\}$ and $\mathscr{V}$. The irreducibility of the decomposition (21) refers to the fact that any representation of the three-dimensional rotation group $SO(3)$ in each of the subspaces $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_3$, $\mathscr{T}_4$ and $\mathscr{T}_5$ is irreducible (Pratz, 1983; Jones, 1985).

Introducing eqn (22) into eqn (18), we get

$$K(\mathbf{N}) = u+\mathbf{U}':\mathbf{N}+\mathbf{N}:\mathbb{K}'\mathbf{N} \tag{23a}$$

$$\kappa(\mathbf{N}) = v+\mathbf{V}':\mathbf{N} \tag{23b}$$

with

$$u = \alpha+2\beta, \qquad v = 3\alpha+2\beta \tag{23c}$$

$$\mathbf{U}' = 2\mathbf{A}'+4\mathbf{B}', \quad \mathbf{V}' = 3\mathbf{A}'+4\mathbf{B}'. \tag{23d}$$

As $\mathbf{U}'$ and $\mathbf{V}'$ are s.t. and $\mathbb{K}'$ is c.s.t. eqns (23a) and (23b) can be written in the equivalent forms

$$K(\mathbf{N}) = u + \mathbf{U}' : \mathbf{F}(\mathbf{N}) + \mathbb{K}' :: \mathbb{F}(\mathbf{N}) \tag{24a}$$

$$\kappa(\mathbf{N}) = v + \mathbf{V}' : \mathbf{F}(\mathbf{N}), \tag{24b}$$

where $\mathbf{F}(\mathbf{N})$ and $\mathbb{F}(\mathbf{N})$ designate the two tensor spherical harmonics given by eqn (8a) and (8b). In fact, equations (24a) and (24b) are the Fourier series expansions of $K(\mathbf{N})$ and $\kappa(\mathbf{N})$, whereby some underlying characteristics of $K(\mathbf{N})$ and $\kappa(\mathbf{N})$ can more easily be seen.

4.3. *A one-to-one correspondence*

Based on eqns (23) or (24), a simple count shows that $K(\mathbf{N})$ depends on 15 independent coefficients while $\kappa(\mathbf{N})$ is determined by six other independent ones. Recall that $\mathbb{K}$ contains 21 independent coefficients. Thus arises naturally the question: does the knowledge of $K(\mathbf{N})$ and $\kappa(\mathbf{N})$ allow us to fully and uniquely determine $\mathbb{K}$? The answer can be formulated as follows.

Theorem 2. Let two functions $K : \mathcal{N} \to \mathcal{R}$ and $\kappa : \mathcal{N} \to \mathcal{R}$ be given by eqns (24a) and (24b). Then there exists a unique fourth-order tensor $\mathbb{K}$ with the minor and major symmetries of eqn (13), such that, for all $\mathbf{N} \in \mathcal{N}$,

$$\text{(i)}\ \ K(\mathbf{N}) = \mathbf{N} : \mathbb{K}\mathbf{N}; \quad \text{(ii)}\ \ \kappa(\mathbf{N}) = \mathbf{I} : \mathbb{K}\mathbf{N}. \qquad \bullet$$

Proof. Existence: given two functions K and κ in the forms (24a) and (24b), we can obtain $u, v, \mathbf{U}', \mathbf{V}'$ and $\mathbb{K}'$ by

$$u = \frac{1}{4\pi}\int_{\mathcal{S}} K(\mathbf{N})\,\mathrm{d}a, \quad v = \frac{1}{4\pi}\int_{\mathcal{S}} \kappa(\mathbf{N})\,\mathrm{d}a \tag{25a}$$

$$\mathbf{U}' = \frac{15}{8\pi}\int_{\mathcal{S}} K(\mathbf{N})\,\mathbf{F}(\mathbf{N})\,\mathrm{d}a, \quad \mathbf{V}' = \frac{15}{8\pi}\int_{\mathcal{S}} \kappa(\mathbf{N})\,\mathbf{F}(\mathbf{N})\,\mathrm{d}a \tag{25b}$$

$$\mathbb{K}' = \frac{315}{32\pi}\int_{\mathcal{S}} K(\mathbf{N})\,\mathbb{F}(\mathbf{N})\,\mathrm{d}a. \tag{25c}$$

Then, define α, β, $\mathbf{A}'$ and $\mathbf{B}'$ as linear combinations of $u, v, \mathbf{U}'$ and $\mathbf{V}'$:

$$\alpha = \tfrac{1}{2}(v-u), \quad \beta = \tfrac{1}{4}(3u-v) \tag{26a}$$

$$\mathbf{A}' = \mathbf{V}' - \mathbf{U}', \quad \mathbf{B}' = \tfrac{1}{4}(3\mathbf{U}' - 2\mathbf{V}'). \tag{26b}$$

In fact, eqns (26a) and (26b) correspond to the inverses of eqns (23c) and (23d). Substituting eqns (26a), (26b) and (25c) into eqn (22) results in

$$\begin{aligned}\mathbb{K} = {} & \tfrac{1}{2}(v-u)\mathbf{I}\otimes\mathbf{I} + \tfrac{1}{4}(3u-v)(\mathbf{I}\,\underline{\otimes}\,\mathbf{I} + \mathbf{I}\,\overline{\otimes}\,\mathbf{I}) + \mathbf{I}\otimes(\mathbf{V}'-\mathbf{U}') \\ & + (\mathbf{V}'-\mathbf{U}')\otimes\mathbf{I} + \tfrac{1}{4}[\mathbf{I}\,\underline{\otimes}\,(3\mathbf{U}'-2\mathbf{V}') + (3\mathbf{U}'-2\mathbf{V}')\,\underline{\otimes}\,\mathbf{I} \\ & + \mathbf{I}\,\overline{\otimes}\,(3\mathbf{U}'-2\mathbf{V}') + (3\mathbf{U}'-2\mathbf{V}')\,\overline{\otimes}\,\mathbf{I}] + \mathbb{K}',\end{aligned} \tag{27}$$

so that conditions (i) and (ii) are satisfied.

Uniqueness (Ladevèze, 1983): suppose there exist two fourth-order tensors $\mathbb{K}$ and $\mathbb{W}$ with minor and major symmetries such that the conditions (i) and (ii) are verified. Setting $\mathbb{M} = \mathbb{K} - \mathbb{W}$, then we have, for every $\mathbf{N} \in \mathcal{N}$,

$$\text{(a) } \mathbf{I} : \mathbb{M}\mathbf{N} = 0; \quad \text{(b) } \mathbf{N} : \mathbb{M}\mathbf{N} = 0.$$

We shall show that these two conditions imply that $\mathbb{M} = \mathbb{O}$, i.e. uniqueness. To do this, we proceed in two steps.

Step 1. Since $\mathbb{M}$ has minor and major symmetries, the second-order tensor $\mathbf{T} := \mathbb{M}\mathbf{I}$ is symmetric and the condition (a) is equivalent to

$$\text{(a}'\text{) } \mathbf{T} : \mathbf{N} = 0$$

for every $\mathbf{N} \in \mathcal{N}$. According to the spectral theorem, any second-order symmetric tensor $\mathbf{X}$ admits the following decomposition:

$$\text{(c) } \mathbf{X} = x_1\mathbf{N}_1 + x_2\mathbf{N}_2 + x_3\mathbf{N}_3,$$

where $\mathbf{N}_i = \mathbf{n}_i \otimes \mathbf{n}_i$ $(i = 1, 2, 3)$, $\mathbf{n}_i$ being the unit eigenvector associated with the eigenvalue x_i of $\mathbf{X}$, such that $\mathbf{n}_i \cdot \mathbf{n}_j = 0$ $(i \neq j)$. Taking (a′) into account while using (c), we deduce that $\mathbf{T} : \mathbf{X} = 0$ for any second order symmetric tensor $\mathbf{X}$ and this means that

$$\text{(a}''\text{) } \mathbb{M}\mathbf{I} = \mathbf{T} = \mathbf{0}.$$

Step 2. Let $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ be three orthonormal vectors. Then,

$$\text{(d) } \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{N}_3 = \mathbf{I},$$

with $\mathbf{N}_i = \mathbf{n}_i \otimes \mathbf{n}_i$ $(i = 1, 2, 3)$. Due to the conditions (b), (d) and (a″),

$$0 = \mathbf{N}_1 : \mathbb{M}\mathbf{N}_1 = \mathbf{N}_1 : \mathbb{M}(\mathbf{I} - \mathbf{N}_2 - \mathbf{N}_3) = -\mathbf{N}_1 : \mathbb{M}\mathbf{N}_2 - \mathbf{N}_1 : \mathbb{M}\mathbf{N}_3$$

i.e.

$$\text{(e) } \mathbf{N}_1 : \mathbb{M}\mathbf{N}_2 = -\mathbf{N}_1 : \mathbb{M}\mathbf{N}_3.$$

Similarly, we can write

$$\text{(f) } \mathbf{N}_1 : \mathbb{M}\mathbf{N}_3 = -\mathbf{N}_2 : \mathbb{M}\mathbf{N}_3, \quad \text{(g) } \mathbf{N}_2 : \mathbb{M}\mathbf{N}_3 = -\mathbf{N}_1 : \mathbb{M}\mathbf{N}_2.$$

Connecting (e), (f) and (g) leads to $\mathbf{N}_i : \mathbb{M}\mathbf{N}_j = 0$ $(i,j = 1, 2, 3; i \neq j)$. More generally,

$$\text{(h) } \mathbf{N}_i : \mathbb{M}\mathbf{N}_j = 0 \quad \text{with} \quad i,j = 1, 2, 3,$$

since, according to (b), $\mathbf{N}_i : \mathbb{M}\mathbf{N}_j = 0$ $(i,j = 1, 2, 3; i = j)$. It follows immediately from (c) and (h) that, for any second-order tensor $\mathbf{X}$,

$$\mathbf{X} : \mathbb{M}\mathbf{X} = 0.$$

With the major symmetry of $\mathbb{M}$, this condition implies that $\mathbb{M} = \mathbb{O}$. ●

From an experimental point of view, Theorem 2 tells us that *pure elongation* tests are sufficient to completely identify the elasticity tensor. In addition, it is clear from eqns (24a) and (24b) that a first-order polynomial in $\mathbf{N}$ is general enough to approximate experimentally obtained data of $\kappa(\mathbf{N})$, while a second-order polynomial in $\mathbf{N}$ is generally needed for approximating those of $K(\mathbf{N})$. When experimental data are limited, eqns (24a) and (24b) also provide the possibility of neglecting only the terms related to higher tensor spherical harmonics. For example, we can use $K(\mathbf{N}) = u + \mathbf{U}' : \mathbf{N}$ and $\kappa(\mathbf{N}) = v$ for averaging certain limited data from pure elongation tests of an orthotropic elastic material, optimize the coefficients u, v and $\mathbf{U}'$ with respect to the norm $L^2(\mathcal{S})$, and then introduce their optimized values into eqn (27) to get an orthotropic elasticity tensor $\mathbb{K}$ depending only on four constants. Such a data-averaging method seems to be able to overcome the difficulties arising from the *redundancy* of experimental data (see Grédiac *et al.*, 1993).

Alternatively, if choosing the stress tensor **S** to be a *controlled* (or independent) variable, we can correspondingly define the *Young and bulk modulus* o.d.f.s by considering *pure traction* tests instead of pure elongation tests. The results established in this section remain usable in the dual sense. From an experimental point of view, this approach is more advantageous, since a pure traction test is much easier to achieve than a pure elongation test. Nevertheless, when a damaged elastic material which may be *softening* is concerned, the strain tensor **E** must be taken as the control variable (see He and Curnier, 1994).

5. DAMAGE VARIABLES AND DAMAGED ELASTIC BEHAVIOR

We now proceed to develop a systematic approach to damaged elastic stress–strain relations. The results obtained in the preceding sections will be employed in both undamaged and damaged cases, provided the necessary conditions are satisfied. A word about the notation is useful: the subscript 0 will refer to the undamaged state while $\tilde{}$ will be assigned to a damage state of the material.

5.1. *A preliminary definition*

Consider a material macroscopic scalar property ω, whose degradation can be used as a *macroscopic indicator* of the underlying microstructural damage processes (Lemaitre and Dufailly, 1987). We denote by ω_0 its value at the *undamaged (or initial) state* and by $\tilde{\omega}$ its value at some later *damaged state* of the material. Both ω_0 and $\tilde{\omega}$ vary generally with the orientation specified by $\mathbf{n}$. Let two functions $w_0\colon \mathscr{S} \to \mathscr{R}$ and $\tilde{w}\colon \mathscr{S} \to \mathscr{R}$ be such as to describe the directional dependences of ω_0 and $\tilde{\omega}$; that is, $\omega_0 = w_0(\mathbf{n})$ and $\tilde{\omega} = \tilde{w}(\mathbf{n})$. Then, the function $\Omega\colon \mathscr{S} \to \mathscr{R}$, defined by

$$\Omega(\mathbf{n}) := \frac{w_0(\mathbf{n}) - \tilde{w}(\mathbf{n})}{w_0(\mathbf{n})} = 1 - \frac{\tilde{w}(\mathbf{n})}{w_0(\mathbf{n})}, \tag{28}$$

can be taken as a measure of the degree of material damage in the direction **n** and will be referred to as a *damage* o.d.f.

Invoking the same argument as used for writing the invariance condition, eqn (3), we have

$$w_0(\mathbf{n}) = w_0(-\mathbf{n}), \quad \tilde{w}(\mathbf{n}) = \tilde{w}(-\mathbf{n}) \tag{29}$$

for $\mathbf{n} \in \mathscr{S}$. It follows from Lemma 1 and eqn (29) that there exist two functions $\hat{w}_0\colon \mathscr{N} \to \mathscr{R}$ and $\hat{w}\colon \mathscr{N} \to \mathscr{R}$ such that

$$w_0(\mathbf{n}) = \hat{w}_0(\mathbf{N}), \quad \tilde{w}(\mathbf{n}) = \hat{w}(\mathbf{N}) \tag{30}$$

for all $\mathbf{n} \in \mathscr{S}$. Introducing eqn (30) into eqn (28) yields

$$\Omega(\mathbf{n}) = \hat{\Omega}(\mathbf{N}) := 1 - \frac{\hat{w}(\mathbf{N})}{\hat{w}_0(\mathbf{N})}, \tag{31}$$

where $\hat{\Omega}\colon \mathscr{N} \to \mathscr{R}$ will also be called a damage o.d.f.

5.2. *The principal hypothesis and damage variables*

In the following we shall be concerned only with the influence of damage on the material elastic response. The undamaged material behavior is assumed to be *linear and hyperelastic*:

$$\mathbf{S} = \mathbb{K}_0\mathbf{E}, \tag{32}$$

where $\mathbb{K}_0$, the *undamaged elasticity tensor*, is positive definite and possesses minor and major symmetries. Apart from eqn (32), the *principal hypothesis* we shall make is that the damaged elastic behavior *at a fixed damage state* is also *linear* and *hyperelastic*:

$$\mathbf{S} = \tilde{\mathbb{K}}\mathbf{E} \tag{33}$$

where the *damaged elasticity tensor* $\tilde{\mathbb{K}}$, a function of the damage variables to be chosen, is also positive definite and has minor and major symmetries. These hypotheses, general enough for many engineering problems, are found in almost all existing *elastic* damage theories.

Next we introduce the *undamaged* and *damaged* elongation and bulk modulus o.d.f.s as follows:

$$K_0(\mathbf{N}) = \mathbf{N}:\mathbb{K}_0\mathbf{N}, \quad \kappa_0(\mathbf{N}) = \mathbf{I}:\mathbb{K}_0\mathbf{N}; \tag{34}$$

$$\tilde{K}(\mathbf{N}) = \mathbf{N}:\tilde{\mathbb{K}}\mathbf{N}, \qquad \tilde{\kappa}(\mathbf{N}) = \mathbf{I}:\tilde{\mathbb{K}}\mathbf{N}. \tag{35}$$

According to Theorem 2, $\mathbb{K}_0$ is uniquely determined by the knowledge of $K_0(\mathbf{N})$ and $\kappa_0(\mathbf{N})$, and $\tilde{\mathbb{K}}$ by that of $\tilde{K}(\mathbf{N})$ and $\tilde{\kappa}(\mathbf{N})$. If the material elasticity degradation is considered as the macroscopic indicator of the underlying microstructural damage processes, then the following *two* damage o.d.f.s

$$\hat{d}(\mathbf{N}) := \frac{K_0(\mathbf{N}) - \tilde{K}(\mathbf{N})}{K_0(\mathbf{N})} = 1 - \frac{\tilde{K}(\mathbf{N})}{K_0(\mathbf{N})}, \tag{36}$$

$$\hat{\eta}(\mathbf{N}) := \frac{\kappa_0(\mathbf{N}) - \tilde{\kappa}(\mathbf{N})}{\kappa_0(\mathbf{N})} = 1 - \frac{\tilde{\kappa}(\mathbf{N})}{\kappa_0(\mathbf{N})} \tag{37}$$

fully characterize the damage state of the material. The fact that *two* scalar-valued functions defined on $\mathscr{S}$ are needed for completely describing the damage state of an elastic linear damageable material was first indicated by Ladevèze (1983).

In order to get a finite characterization of a given damage state, we assume $\hat{d}(\mathbf{N})$ and $\hat{\eta}(\mathbf{N})$ to be square-integrable and expand them in the convergent Fourier series:

$$\hat{d}(\mathbf{N}) = \delta + \mathbf{D}':\mathbf{F}(\mathbf{N}) + \mathbb{D}'::\mathbb{F}(\mathbf{N}) + \cdots, \tag{38}$$

$$\hat{\eta}(\mathbf{N}) = h + \mathbf{H}':\mathbf{F}(\mathbf{N}) + \mathbb{H}'::\mathbb{F}(\mathbf{N}) + \cdots, \tag{39}$$

where δ and h are two scalars, $\mathbf{D}'$ and $\mathbf{H}'$ two s.t. second-order tensors, and $\mathbb{D}'$ and $\mathbb{H}'$ two c.s.t. fourth-order tensors. These expansion coefficients can naturally be taken as *damage variables*. Hence, a finite description of the damage state amounts to using a limited number of expansion coefficients, for example, $\{\delta, \mathbf{D}', \mathbb{D}'\}$ and $\{h, \mathbf{H}'\}$, as the damage variables on which $\tilde{\mathbb{K}}$ depends.

5.3. *Damaged elasticity tensor*

Let the undamaged elasticity tensor $\mathbb{K}_0$ be given. Following the procedure presented in Section 4.2, we have the irreducible decomposition of $\mathbb{K}_0$:

$$\mathbb{K}_0 = \alpha_0\,\mathbf{I}\otimes\mathbf{I} + \beta_0\,(\mathbf{I}\,\underline{\otimes}\,\mathbf{I} + \mathbf{I}\,\bar{\otimes}\,\mathbf{I}) + \mathbf{I}\otimes\mathbf{A}'_0 + \mathbf{A}'_0\otimes\mathbf{I} + \mathbf{I}\,\underline{\otimes}\,\mathbf{B}'_0 + \mathbf{B}'_0\,\underline{\otimes}\,\mathbf{I} + \mathbf{I}\,\bar{\otimes}\,\mathbf{B}'_0 + \mathbf{B}'_0\,\bar{\otimes}\,\mathbf{I} + \mathbb{K}'_0 \tag{40}$$

and the Fourier series expansions of $K_0(\mathbf{N})$ and $\kappa_0(\mathbf{N})$:

$$K_0(\mathbf{N}) = u_0 + \mathbf{U}'_0 : \mathbf{N} + \mathbf{N} : \mathbb{K}'_0 \mathbf{N} \tag{41a}$$

$$\kappa_0(\mathbf{N}) = v_0 + \mathbf{V}'_0 : \mathbf{N}, \tag{41b}$$

with

$$u_0 = \alpha_0 + 2\beta_0, \qquad v_0 = 3\alpha_0 + 2\beta_0 \tag{41c}$$

$$\mathbf{U}'_0 = 2\mathbf{A}'_0 + 4\mathbf{B}'_0, \quad \mathbf{V}'_0 = 3\mathbf{A}'_0 + 4\mathbf{B}'_0. \tag{41d}$$

Expressions similar to eqns (40) and (41) exist for $\tilde{\mathbb{K}}$. However, for the purpose of this paragraph it suffices to write

$$\tilde{K}(\mathbf{N}) = \tilde{u} + \tilde{\mathbf{U}}' : \mathbf{N} + \mathbf{N} : \tilde{\mathbb{K}}' \mathbf{N} \tag{42a}$$

$$\tilde{\kappa}(\mathbf{N}) = \tilde{v} + \tilde{\mathbf{V}}' : \mathbf{N}. \tag{42b}$$

On the other hand, it is immediate from definitions (36) and (37) that

$$\tilde{K}(\mathbf{N}) = \bar{K}(\mathbf{N}), \quad \tilde{\kappa}(\mathbf{N}) = \bar{\kappa}(\mathbf{N}) \tag{43}$$

with

$$\bar{K}(\mathbf{N}) := [1 - \hat{d}(\mathbf{N})]\, K_0(\mathbf{N}), \quad \bar{\kappa}(\mathbf{N}) := [1 - \hat{\eta}(\mathbf{N})]\, \kappa_0(\mathbf{N}). \tag{44}$$

Introducing eqns (38), (39), (41a) and (41b) into eqn (44), yields

$$\begin{aligned} \bar{K}(\mathbf{N}) = {} & (1-\delta)u_0 + [(1-\delta)\mathbf{U}'_0 - u_0 \mathbf{D}'] : \mathbf{F}(\mathbf{N}) + [(1-\delta)\mathbb{K}'_0 - u_0 \mathbb{D}'] :: \mathbb{F}(\mathbf{N}) \\ & - (\mathbf{D}' \otimes \mathbf{U}'_0) :: (\mathbf{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N})) - (\mathbb{K}'_0 \otimes \mathbf{D}') ::: (\mathbb{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N})) \\ & - (\mathbb{D}' \otimes \mathbf{U}'_0) ::: (\mathbb{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N})) - (\mathbb{D}' \otimes \mathbb{K}'_0) :::: (\mathbb{F}(\mathbf{N}) \otimes \mathbb{F}(\mathbf{N})) + \cdots \end{aligned} \tag{45a}$$

$$\begin{aligned} \bar{\kappa}(\mathbf{N}) = {} & (1-h)v_0 + [(1-h)\mathbf{V}'_0 - v_0 \mathbf{H}'] : \mathbf{F}(\mathbf{N}) - v_0 \mathbb{H}' :: \mathbb{F}(\mathbf{N}) \\ & - (\mathbf{H}' \otimes \mathbf{V}'_0) :: (\mathbf{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N})) - (\mathbb{H}' \otimes \mathbf{V}'_0) ::: (\mathbb{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N})) + \cdots \end{aligned} \tag{45b}$$

By eqns (43) and (42),

$$\tilde{u} + \tilde{\mathbf{U}}' : \mathbf{F}(\mathbf{N}) + \tilde{\mathbb{K}}' :: \mathbb{F}(\mathbf{N}) = \bar{K}(\mathbf{N}) \tag{46}$$

$$\tilde{v} + \tilde{\mathbf{V}}' : \mathbf{F}(\mathbf{N}) = \bar{\kappa}(\mathbf{N}). \tag{47}$$

Then, as a result of the orthogonality properties, eqn (9), of $\{1, \mathbf{F}(\mathbf{N}), \mathbb{F}(\mathbf{N}), \ldots\}$, we have

$$\begin{aligned} \tilde{u} &= \frac{1}{4\pi} \int_{\mathscr{S}} \bar{K}(\mathbf{N})\, \mathrm{d}a, \quad \tilde{\mathbf{U}}' = \frac{15}{8\pi} \int_{\mathscr{S}} \bar{K}(\mathbf{N})\, \mathbf{F}(\mathbf{N})\, \mathrm{d}a \\ \tilde{v} &= \frac{1}{4\pi} \int_{\mathscr{S}} \bar{\kappa}(\mathbf{N})\, \mathrm{d}a, \quad \tilde{\mathbf{V}}' = \frac{15}{8\pi} \int_{\mathscr{S}} \bar{\kappa}(\mathbf{N})\, \mathbf{F}(\mathbf{N})\, \mathrm{d}a \\ \tilde{\mathbb{K}}' &= \frac{315}{32\pi} \int_{\mathscr{S}} \bar{K}(\mathbf{N}) \mathbb{F}(\mathbf{N})\, \mathrm{d}a, \end{aligned} \tag{48}$$

in which the expressions of $\bar{K}(\mathbf{N})$ and $\bar{\kappa}(\mathbf{N})$ are given by eqn (45). When the undamaged elastic behavior, i.e. $\{u_0, \mathbf{U}'_0, \mathbb{K}'_0\}$ and $\{v_0, \mathbf{V}'_0\}$, is specified and a choice of a finite number of damage variables among $\{\delta, h, \mathbf{D}', \mathbf{H}', \mathbb{D}', \mathbb{H}', \ldots\}$ is made, the integrals of eqn (48) can explicitly be carried out. Several important examples will be treated in Sections 6 and 7.

We are now in a position to give the general expression of the damaged elasticity tensor. Indeed, following Theorem 2, a unique $\tilde{\mathbb{K}}$ is associated with $\bar{K}(\mathbf{N})$ and $\bar{\kappa}(\mathbf{N})$ and has the form

$$\mathbb{K} = \tfrac{1}{2}(\tilde{v}-\tilde{u})\,\mathbf{I}\otimes\mathbf{I}+\tfrac{1}{4}(3\tilde{u}-\tilde{v})(\mathbf{I}\,\underline{\otimes}\,\mathbf{I}+\mathbf{I}\,\bar{\otimes}\,\mathbf{I})+\mathbf{I}\otimes(\tilde{\mathbf{V}}'-\tilde{\mathbf{U}}')$$
$$+(\tilde{\mathbf{V}}'-\tilde{\mathbf{U}}')\otimes\mathbf{I}+\tfrac{1}{4}[\mathbf{I}\,\underline{\otimes}\,(3\tilde{\mathbf{U}}'-2\tilde{\mathbf{V}}')+(3\tilde{\mathbf{U}}'-2\tilde{\mathbf{V}}')\,\underline{\otimes}\,\mathbf{I}$$
$$+\mathbf{I}\,\bar{\otimes}\,(3\tilde{\mathbf{U}}'-2\tilde{\mathbf{V}}')+(3\tilde{\mathbf{U}}'-2\tilde{\mathbf{V}}')\,\bar{\otimes}\,\mathbf{I}]+\tilde{\mathbb{K}}', \quad (49)$$

which comes from the application of formula (27) to the damaged case. Recall that $\tilde{u}$, $\tilde{v}$, $\tilde{\mathbf{U}}'$, $\tilde{\mathbf{V}}'$ and $\tilde{\mathbb{K}}'$, calculated by eqn (48), depend on the undamaged elastic properties $\{u_0, \mathbf{U}'_0, \mathbb{K}'_0\}$ and $\{v_0, \mathbf{V}'_0\}$ and on the damage variables $\{\delta, h, \mathbf{D}', \mathbf{H}', \mathbb{D}', \mathbb{H}', \ldots\}$.

The damaged elastic response in the most widely used theories of continuum damage mechanics (Lemaitre and Chaboche, 1978; Cordebois and Sidoroff, 1979; Lemaitre and Chaboche, 1985; Simo and Ju, 1987; Chow and Wang, 1987) is formulated through the concept of effective stress and the hypothesis of strain or stress-energy equivalence. As can be seen, our approach is quite different from the preceding ones; it starts from the definition of two natural damage o.d.f.s and then resorts to Theorem 2 for obtaining the damaged elasticity tensor. Thus, certain drawbacks arising from the lack of uniformity and rigor (see Rabier, 1989, for details) of the existing elastic damage theories are remedied.

5.4. *Restrictions on the choice of damage variables*

It is important to remark that, for eqns (46) and (47) to hold, certain restrictive conditions must be verified by the damage variables. As a matter of fact, eqns (46) and (47) imply that

$$\int_{\mathscr{S}} \bar{K}(\mathbf{N})\,\mathbb{F}_m(\mathbf{N})\,\mathrm{d}a = \mathbb{O}_m, \quad (m = 6, 8, 10, \ldots) \quad (50a)$$

$$\int_{\mathscr{S}} \bar{\kappa}(\mathbf{N})\,\mathbb{F}_m(\mathbf{N})\,\mathrm{d}a = \mathbb{O}_m, \quad (m = 4, 6, 8, \ldots), \quad (50b)$$

where $\mathbb{F}_m(\mathbf{N})$ denotes the mth-order tensor spherical harmonic (see e.g. Kanatani, 1984 for its exact form) and $\mathbb{O}_m$ the mth-order zero tensor. Inserting eqn (45) into eqn (50) gives the conditions to be satisfied by $\{\delta, \mathbf{D}', \mathbb{D}', \ldots\}$ and $\{h, \mathbf{H}', \mathbb{H}', \ldots\}$. Such conditions should not surprise us, since they result from the principal hypothesis, eqn (33), that the stress–strain relation at a fixed damage state is *linear*.

If the considered material is *initially isotropic*, then eqns (41a) and (41b) become very simple:

$$K_0(\mathbf{N}) = u_0, \quad \kappa_0(\mathbf{N}) = v_0 \quad (51)$$

and eqns (45a) and (45b) reduce to

$$\bar{K}(\mathbf{N}) = u_0(1-\delta)-u_0\mathbf{D}':\mathbf{F}(\mathbf{N})-u_0\mathbb{D}'::\mathbb{F}(\mathbf{N})+\cdots,$$
$$\bar{\kappa}(\mathbf{N}) = v_0(1-h)-v_0\mathbf{H}':\mathbf{F}(\mathbf{N})+\cdots.$$

Then it is not difficult to show that, in such a case, eqns (50a) and (50b) are satisfied if and only if

$$\bar{K}(\mathbf{N}) = u_0(1-\delta)-u_0\mathbf{D}':\mathbf{F}(\mathbf{N})-u_0\mathbb{D}'::\mathbb{F}(\mathbf{N}), \quad (52a)$$

$$\bar{\kappa}(\mathbf{N}) = v_0(1-h)-v_0\mathbf{H}':\mathbf{F}(\mathbf{N}). \quad (52b)$$

This implies that, within the framework consistent with hypothesis (33), the most general forms of $\hat{d}(\mathbf{N})$ and $\hat{\eta}(\mathbf{N})$ are

$$\hat{d}(\mathbf{N}) = \delta + \mathbf{D}' : \mathbf{F}(\mathbf{N}) + \mathbb{D}' :: \mathbb{F}(\mathbf{N}), \tag{53a}$$

$$\hat{\eta}(\mathbf{N}) = h + \mathbf{H}' : \mathbf{F}(\mathbf{N}). \tag{53b}$$

In other words, the set $\{\delta, h, \mathbf{D}', \mathbf{H}', \mathbb{D}'\}$ (or a fourth-order tensor with the minor and major symmetries) constitutes the most general choice of damage variables. This conclusion is not surprising and corresponds to our intuition.

However, when the material is *initially anisotropic*, it is much more difficult to choose, without loss of generality, damage variables such as to verify conditions (50). Suppose a choice of damage variables, guided by some available experimental observations and analyses, is made and conditions (50a) and (50b) are not exactly satisfied. If the values of the two integrals

$$\varepsilon_K = \int_{\mathscr{S}} |\bar{K}(\mathbf{N}) - \tilde{u} - \tilde{\mathbf{U}}' : \mathbf{F}(\mathbf{N}) - \tilde{\mathbb{K}}' :: \mathbb{F}(\mathbf{N})|^2 \, da \tag{54a}$$

$$\varepsilon_\kappa = \int_{\mathscr{S}} |\bar{\kappa}(\mathbf{N}) - \tilde{v} - \tilde{\mathbf{V}}' : \mathbf{N}|^2 \, da \tag{54b}$$

in which $\{\tilde{u}, \tilde{\mathbf{U}}', \tilde{\mathbb{K}}'\}$ and $\{\tilde{v}, \tilde{\mathbf{V}}'\}$ are obtained by eqn (48), are such that

$$\varepsilon_K \ll \int_{\mathscr{S}} |\bar{K}(\mathbf{N})|^2 \, da, \quad \varepsilon_\kappa \ll \int_{\mathscr{S}} |\bar{\kappa}(\mathbf{N})|^2 \, da, \tag{55}$$

then conditions (50a) and (50b) are said to be *approximately satisfied in the mean*. In this sense, we write

$$\bar{K}(\mathbf{N}) \approx \tilde{u} + \tilde{\mathbf{U}}' : \mathbf{F}(\mathbf{N}) + \tilde{\mathbb{K}}' :: \mathbb{F}(\mathbf{N}) \tag{56a}$$

$$\bar{\kappa}(\mathbf{N}) \approx \tilde{v} + \tilde{\mathbf{V}}' : \mathbf{F}(\mathbf{N}), \tag{56b}$$

and replace $=$ by $\approx$ in formula (49). We shall, in Section 7, establish a damaged elastic stress–strain relation for unidirectional fibre-reinforced composites in this way.

6. APPLICATION TO INITIALLY ISOTROPIC MATERIALS

We recall that the undamaged elongation and bulk modulus o.d.f.s $K_0(\mathbf{N})$ and $\kappa_0(\mathbf{N})$ for initially isotropic materials are specified by eqn (51), where, if necessary, u_0 and v_0 can be expressed in terms of Young's modulus E_0 and Poisson's ration ν_0 by means of eqn (19). In addition, the general forms of $\bar{K}(\mathbf{N})$, $\bar{\kappa}(\mathbf{N})$, $\hat{d}(\mathbf{N})$ and $\hat{\eta}(\mathbf{N})$ are given in eqns (52a)–(53b).

By applying eqn (48) to eqn (52), we get

$$\tilde{u} = u_0(1-\delta), \quad \tilde{\mathbf{U}}' = -u_0\mathbf{D}', \quad \tilde{\mathbb{K}}' = -u_0\mathbb{D}'; \tag{57a}$$

$$\tilde{v} = v_0(1-h), \quad \tilde{\mathbf{V}}' = -v_0\mathbf{H}'. \tag{57b}$$

These expressions can directly be obtained by comparing eqns (52a) and (52b), respectively, with eqns (46) and (47). Then, introducing eqn (57) into eqn (49) yields the general expression of $\tilde{\mathbb{K}}$ in terms of the undamaged isotropic elastic constants $\{u_0, v_0\}$ and the damage variables $\{\delta, h, \mathbf{D}', \mathbf{H}', \mathbb{D}'\}$.

The aforementioned formulation is too general to be practicable. Indeed, given u_0 and v_0, it amounts to choosing the fourth-order elasticity tensor $\mathbb{K}$ as the internal variable and hence obliges us to establish the evolution law for $\mathbb{K}$, which is tremendously difficult. A useful elastic damage model should be able to reflect the main effects of damage on the elastic properties with a reasonable number of internal variables. In fact, the approach developed in Section 5 is particularly adapted to such a demand, since it presents the possibility of neglecting, in eqns (53a) and (53b) or more generally in eqns (38) and (39), the terms which are related to high-order spherical harmonics and thus represent low-order effects of damage on the elongation and bulk moduli. We now deal with two important examples.

6.1. *Isotropic damage*

If the following invariance conditions

$$\hat{d}(\mathbf{QNQ}^{\mathrm{T}}) = \hat{d}(\mathbf{N}), \quad \hat{\eta}(\mathbf{QNQ}^{\mathrm{T}}) = \hat{\eta}(\mathbf{N}), \quad \forall \mathbf{Q} \in SO(3) \tag{58}$$

are satisfied, then the damage is isotropic and eqns (53a) and (53b) reduce to

$$\hat{d}(\mathbf{N}) = \delta, \quad \hat{\eta}(\mathbf{N}) = h, \tag{59}$$

which corresponds to the zero-order damage kinematics of Ladevèze (1983, 1993). In agreement with eqn (59),

$$\tilde{K}(\mathbf{N}) = \tilde{u} = (1-\delta)u_0, \quad \tilde{\kappa}(\mathbf{N}) = \tilde{v} = (1-h)v_0. \tag{60}$$

By eqn (49), we get

$$\tilde{\mathbb{K}} = \tfrac{1}{2}[(1-h)v_0-(1-\delta)u_0]\mathbf{I}\otimes\mathbf{I}+\tfrac{1}{4}[3(1-\delta)u_0-(1-h)v_0](\mathbf{I}\underline{\otimes}\mathbf{I}+\mathbf{I}\overline{\otimes}\mathbf{I}). \tag{61}$$

As the damaged elastic material remains isotropic, we can use formula (19) to express $\tilde{K}(\mathbf{N})$ and $\tilde{\kappa}(\mathbf{N})$ in terms of the corresponding damaged Young's modulus $\tilde{E}$ and Poisson's ratio $\tilde{\nu}$,

$$\tilde{K}(\mathbf{N}) = \frac{(1-\tilde{\nu})\tilde{E}}{(1+\tilde{\nu})(1-2\tilde{\nu})}, \quad \tilde{\kappa}(\mathbf{N}) = \frac{\tilde{E}}{1-2\tilde{\nu}} \tag{62}$$

and to write eqn (60) in the following equivalent forms:

$$\tilde{E} = \frac{3(1-\delta)(1-h)(1-\nu_0)-(1-h)^2(1+\nu_0)}{(1-h)(1+\nu_0)(1-2\nu_0)+(1-\delta)(1-\nu_0)(1-2\nu_0)}E_0, \tag{63a}$$

$$\tilde{\nu} = \frac{(1-h)(1+\nu_0)-(1-\delta)(1-\nu_0)}{(1-h)(1+\nu_0)+(1-\delta)(1-\nu_0)}. \tag{63b}$$

The two latter expressions, which relate $\tilde{E}$ and $\tilde{\nu}$ to E_0 and ν_0 through δ and h, are not simple. This is due to the fact that the strain tensor is chosen to be the controlled variable and that pure elongation tests, instead of pure traction ones, are privileged. Such a strain-controlled damage formulation is more advantageous than the stress-controlled one, since it is more convenient for the variational formulation of a boundary value problem and it describes a wider class of materials when damage evolution equations are established within the framework of generalized standard materials (He and Curnier, 1994).

It is seen from eqn (59) that a general description of isotropic elastic damage necessitates *two* scalar internal variables instead of one as in many isotropic damage theories. This has been pointed out by Ladevèze (1983), Rabier (1989) and Ju (1990).

6.2. *Orthotropic damage*

Damage processes are generally anisotropic and often exhibit privileged directions. In the case of initially isotropic materials subjected to *proportional loading*, it is reasonable to make the assumption that damage is *orthotropic* with respect to the principal axes of the strain tensor. Under this hypothesis, we now apply the developed approach to the construction of two orthotropic damage models.

Let $\mathbf{D}$ and $\mathbf{H}$ be two second-order tensors which are *symmetric and positive semi-definite*. Suppose the damage o.d.f.s $\hat{d}(\mathbf{N})$ and $\hat{\eta}(\mathbf{N})$ are such that

$$\hat{d}(\mathbf{N}) = \delta + \mathbf{D}' : \mathbf{F}(\mathbf{N}), \tag{64a}$$

$$\hat{\eta}(\mathbf{N}) = h + \mathbf{H}' : \mathbf{F}(\mathbf{N}), \tag{64b}$$

with

$$\delta = \tfrac{1}{3}\operatorname{tr}\mathbf{D}, \quad \mathbf{D}' = \mathbf{D} - \delta\mathbf{I}; \quad h = \tfrac{1}{3}\operatorname{tr}\mathbf{H}, \quad \mathbf{H}' = \mathbf{H} - h\mathbf{I}. \tag{64c}$$

This assumption corresponds to the first-order damage kinematics of Ladevèze (1983, 1993). Comparing eqn (64a) with eqn (53a), we see that only the term related to the fourth-order harmonic $\mathbb{F}(\mathbf{N})$ is neglected. With eqn (64), $\bar{K}(\mathbf{N})$ and $\bar{\kappa}(\mathbf{N})$ given by eqns (52a) and (52b) become

$$\bar{K}(\mathbf{N}) = \tilde{u} + \tilde{\mathbf{U}}' : \mathbf{F}(\mathbf{N}), \quad \tilde{u} = (1-\delta)u_0, \quad \tilde{\mathbf{U}}' = -u_0\mathbf{D}' \tag{65a}$$

$$\bar{\kappa}(\mathbf{N}) = \tilde{v} + \tilde{\mathbf{V}}' : \mathbf{F}(\mathbf{N}), \quad \tilde{v} = (1-h)v_0, \quad \tilde{\mathbf{V}}' = -u_0\mathbf{H}'. \tag{65b}$$

By eqn (49), we obtain the associated damaged elasticity tensor

$$\begin{aligned}\tilde{\mathbb{K}} = {} & \tfrac{1}{2}[(1-h)v_0 - (1-\delta)u_0]\mathbf{I} \otimes \mathbf{I} + \tfrac{1}{4}[3(1-\delta)u_0 - (1-h)v_0](\mathbf{I} \underline{\otimes} \mathbf{I} + \mathbf{I} \bar{\otimes} \mathbf{I}) \\ & - u_0\mathbf{I} \otimes (\mathbf{H}' - \mathbf{D}') - u_0(\mathbf{H}' - \mathbf{D}') \otimes \mathbf{I} - \frac{u_0}{4}[\mathbf{I} \underline{\otimes} (3\mathbf{D}' - 2\mathbf{H}') \\ & + (3\mathbf{D}' - 2\mathbf{H}') \underline{\otimes} \mathbf{I} + \mathbf{I} \bar{\otimes} (3\mathbf{D}' - 2\mathbf{H}') + (3\mathbf{D}' - 2\mathbf{H}) \bar{\otimes} \mathbf{I}].\end{aligned} \tag{66}$$

In general, this tensor is *triclinic* or *fully anisotropic*. A necessary and sufficient condition for it to be *orthotropic* is that the two damage tensors $\mathbf{D}$ and $\mathbf{H}$ be *commutative*:

$$\mathbf{DH} = \mathbf{HD} \tag{67}$$

or equivalently *coaxial*:

$$\mathbf{D} = D_1\mathbf{d}_1 \otimes \mathbf{d}_1 + D_2\mathbf{d}_2 \otimes \mathbf{d}_2 + D_3\mathbf{d}_3 \otimes \mathbf{d}_3 \tag{68a}$$

$$\mathbf{H} = H_1\mathbf{d}_1 \otimes \mathbf{d}_1 + H_2\mathbf{d}_2 \otimes \mathbf{d}_2 + H_3\mathbf{d}_3 \otimes \mathbf{d}_3, \tag{68b}$$

where D_1, D_2 and D_3 are the eigenvalues of $\mathbf{D}$, H_1, H_2 and H_3 the eigenvalues of $\mathbf{H}$, and $\mathbf{d}_1$, $\mathbf{d}_2$ and $\mathbf{d}_3$, three associated orthonormal eigenvectors of $\mathbf{D}$ (or $\mathbf{H}$). Under the condition (67) or (68), $\tilde{\mathbb{K}}$ becomes *orthotropic* with respect to the three axes defined by $\mathbf{d}_1$, $\mathbf{d}_2$ and $\mathbf{d}_3$.

It is useful to write $\tilde{\mathbb{K}}$, given by eqn (66) together with eqn (68), in matrix form. First, let us associate with $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ the orthonormal basis

$$\begin{aligned}\mathbf{D}_1 &= \mathbf{d}_1 \otimes \mathbf{d}_1, \quad \mathbf{D}_4 = \frac{1}{\sqrt{2}}(\mathbf{d}_2 \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_2) \\ \mathbf{D}_2 &= \mathbf{d}_2 \otimes \mathbf{d}_2, \quad \mathbf{D}_5 = \frac{1}{\sqrt{2}}(\mathbf{d}_3 \otimes \mathbf{d}_1 + \mathbf{d}_1 \otimes \mathbf{d}_3) \\ \mathbf{D}_3 &= \mathbf{d}_3 \otimes \mathbf{d}_3, \quad \mathbf{D}_6 = \frac{1}{\sqrt{2}}(\mathbf{d}_1 \otimes \mathbf{d}_2 + \mathbf{d}_2 \otimes \mathbf{d}_1)\end{aligned} \tag{69}$$

for the space $\mathscr{T}$ of three-dimensional *symmetric* tensors. Then, the strain tensor $\mathbf{E}$, the stress tensor $\mathbf{S}$ and the damaged elasticity tensor $\tilde{\mathbb{K}}$ can be written as

$$\mathbf{E} = \sum_{i=1}^{6} E_i \mathbf{D}_i, \quad \mathbf{S} = \sum_{i=1}^{6} S_i \mathbf{D}_i, \quad \tilde{\mathbb{K}} = \sum_{i,j=1}^{6} \tilde{K}_{ij} \mathbf{D}_i \otimes \mathbf{D}_j, \tag{70}$$

where the components E_i, S_i and $\tilde{K}_{ij}$ are calculated by means of $E_i = \mathbf{E} : \mathbf{D}_i$, $\mathbf{S}_i = \mathbf{S} : \mathbf{D}_i$ and $\tilde{K}_{ij} = \mathbf{D}_i : \tilde{\mathbb{K}} \mathbf{D}_j$. Next the aforementioned orthotropic damaged stress–strain relation has the following matrix form:

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} & \tilde{K}_{13} & & & \\ \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} & & & \\ \tilde{K}_{31} & \tilde{K}_{32} & \tilde{K}_{33} & & & \\ & & & \tilde{K}_{44} & & \\ & & & & \tilde{K}_{55} & \\ & & & & & \tilde{K}_{66} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{bmatrix} \tag{71}$$

with

$$\begin{aligned}
\tilde{K}_{11} &= (1-D_1)u_0, \quad \tilde{K}_{22} = (1-D_2)u_0, \quad \tilde{K}_{33} = (1-D_3)u_0 \\
\tilde{K}_{44} &= \tfrac{1}{2}(3+D_1-2D_2-2D_3)u_0 - \tfrac{1}{2}(1+H_1-H_2-H_3)v_0 \\
\tilde{K}_{55} &= \tfrac{1}{2}(3-2D_1+D_2-2D_3)u_0 - \tfrac{1}{2}(1-H_1+H_2-H_3)v_0 \\
\tilde{K}_{66} &= \tfrac{1}{2}(3-2D_1-2D_2+D_3)u_0 - \tfrac{1}{2}(1-H_1-H_2+H_3)v_0 \\
\tilde{K}_{12} &= \tilde{K}_{21} = \tfrac{1}{2}(1-H_1-H_2+H_3)v_0 - \tfrac{1}{6}(3+D_1+D_2-5D_3)u_0 \\
\tilde{K}_{13} &= \tilde{K}_{31} = \tfrac{1}{2}(1-H_1+H_2-H_3)v_0 - \tfrac{1}{6}(3+D_1-5D_2+D_3)u_0 \\
\tilde{K}_{23} &= \tilde{K}_{32} = \tfrac{1}{2}(1+H_1-H_2-H_3)v_0 - \tfrac{1}{6}(3-5D_1+D_2+D_3)u_0.
\end{aligned} \tag{72}$$

It is seen that, given u_0 and v_0, the matrix of $\mathbb{K}$ contains only six independent parameters: three eigenvalues D_1, D_2 and D_3 of $\mathbf{D}$ plus three eigenvalues H_1, H_2 and H_3 of $\mathbf{H}$. Recall that a general orthotropic elasticity matrix depends on nine parameters. Hence, the stress–strain relation (72) is a particular orthotropic one. The restrictions displayed by eqn (72) will be examined in a future paper.

The most widely used *orthotropic* damage model is probably that proposed by Cordebois and Sidoroff (1979), in which the orthotropic damaged strain–stress relation depends only on three parameters. Compared with eqn (72), their model exhibits more restrictive limitations on the damaged Poisson's ratios.

It is interesting to simplify the previously established orthotropic damage model by setting

$$\mathbf{H} = h\mathbf{I}, \tag{73}$$

which is equivalent to assuming $\hat{\eta}(\mathbf{N})$ to be isotropic, i.e.

$$\hat{\eta}(\mathbf{N}) = h. \tag{74}$$

This simplification is justified by the observation that, in the general expressions of eqn (53), a second-order polynomial in $\mathbf{N}$ is needed for characterizing $\hat{d}(\mathbf{N})$, while a first-order polynomial in $\mathbf{N}$ is sufficient for describing $\hat{\eta}(\mathbf{N})$. In this sense, the effect of damage on the elongation modulus is more anisotropic than on the bulk modulus, and eqn (74) is "consistent" with eqn (64).

The simplified orthotropic damage model depends only on four parameters: h, D_1, D_2 and D_3. More precisely, eqns (71) and (72) remain valid, provided the last six relations of (72) are replaced by

$$\tilde{K}_{44} = \tfrac{1}{2}(3+D_1-2D_2-2D_3)u_0 - \tfrac{1}{2}(1-h)v_0$$
$$\tilde{K}_{55} = \tfrac{1}{2}(3-2D_1+D_2-2D_3)u_0 - \tfrac{1}{2}(1-h)v_0$$
$$\tilde{K}_{66} = \tfrac{1}{2}(3-2D_1-2D_2+D_3)u_0 - \tfrac{1}{2}(1-h)v_0$$
$$\tilde{K}_{12} = \tilde{K}_{21} = \tfrac{1}{2}(1-h)v_0 - \tfrac{1}{6}(3+D_1+D_2-5D_3)u_0$$
$$\tilde{K}_{13} = \tilde{K}_{31} = \tfrac{1}{2}(1-h)v_0 - \tfrac{1}{6}(3+D_1-5D_2+D_3)u_0$$
$$\tilde{K}_{23} = \tilde{K}_{32} = \tfrac{1}{2}(1-h)v_0 - \tfrac{1}{6}(3-5D_1+D_2+D_3)u_0. \quad (75)$$

Clearly, this orthotropic damage model is less general. But it has the advantage that condition (67) is trivially verified by eqn (73), so that eqn (67) imposes no restrictive condition on the evolution equations for **D** and **H**.

7. APPLICATION TO UNIDIRECTIONAL FIBRE-REINFORCED COMPOSITES

The approach developed in Sections 3–5 is also applicable to initially anisotropic materials. To illustrate this, we treat below the problem of formulating the damaged elastic behavior of *unidirectional fibre-reinforced composites.*

7.1. *Undamaged elasticity tensor and its irreducible decomposition*

In many practical cases, unidirectional fibre-reinforced composites can be considered as being *transversely isotropic* at a macroscopic scale. Let the unit vector **m** specify the fibre axis. Then $\mathbf{M} := \mathbf{m} \otimes \mathbf{m}$, called the *structural tensor*, is such that

$$\mathscr{G} := \{\mathbf{Q} \in \mathscr{O}(3) \mid \mathbf{QMQ}^{\mathrm{T}} = \mathbf{M}\}$$

corresponds to the material symmetry group.

The undamaged behavior is assumed to be linear and hyperelastic, so that the undamaged stress–strain relation is given by eqn (32). Due to the introduction of **M**, the undamaged elasticity tensor $\mathbb{K}_0$ can be written in the following invariant form

$$\mathbb{K}_0 = \alpha_1 \mathbf{I} \otimes \mathbf{I} + \alpha_2(\mathbf{I} \underline{\otimes} \mathbf{I} + \mathbf{I} \bar{\otimes} \mathbf{I}) + \alpha_3 \mathbf{M} \otimes \mathbf{M} + \alpha_4(\mathbf{I} \otimes \mathbf{M} + \mathbf{M} \otimes \mathbf{I}) + \alpha_5(\mathbf{I} \underline{\otimes} \mathbf{M} + \mathbf{M} \underline{\otimes} \mathbf{I} + \mathbf{I} \bar{\otimes} \mathbf{M} + \mathbf{I} \bar{\otimes} \mathbf{M}), \quad (76)$$

where α_i $(i = 1, \ldots, 5)$ are five material constants.

The irreducible decomposition of the preceding transversely isotropic elasticity tensor $\mathbb{K}_0$ is given by eqn (40) with

$$\alpha_0 = \tfrac{1}{15}(15\alpha_1 + \alpha_3 + 10\alpha_4), \quad \beta_0 = \tfrac{1}{30}(30\alpha_2 + 2\alpha_3 + 20\alpha_5) \quad (77a)$$
$$\mathbf{A}'_0 = -\tfrac{1}{21}(\alpha_3 + 7\alpha_4)\mathbf{I} + \tfrac{1}{7}(\alpha_3 + 7\alpha_4)\mathbf{M} \quad (77b)$$
$$\mathbf{B}'_0 = -\tfrac{1}{21}(\alpha_3 + 7\alpha_5)\mathbf{I} + \tfrac{1}{7}(\alpha_3 + 7\alpha_5)\mathbf{M} \quad (77c)$$
$$\mathbb{K}'_0 = \alpha_3[\mathbf{M} \otimes \mathbf{M} - \tfrac{1}{7}(\mathbf{I} \otimes \mathbf{M} + \mathbf{M} \otimes \mathbf{I} + \mathbf{I} \underline{\otimes} \mathbf{M} + \mathbf{M} \underline{\otimes} \mathbf{I} + \mathbf{I} \bar{\otimes} \mathbf{M} + \mathbf{M} \bar{\otimes} \mathbf{I}) + \tfrac{1}{35}(\mathbf{I} \otimes \mathbf{I} + \mathbf{I} \underline{\otimes} \mathbf{I} + \mathbf{I} \bar{\otimes} \mathbf{I})]. \quad (77d)$$

How to obtain these expressions from eqn (76) is detailed in Appendix 1. It is worth noting that $\mathbb{K}'_0$ depends only on *one* material constant α_3 and moreover that

$$\mathbb{K}'_0 = \alpha_3 \mathbb{F}(\mathbf{M}), \quad (78)$$

where $\mathbb{F}(.)$ is the fourth-order tensor spherical harmonic function defined by eqn (8b).

The expansions (41a) and (42b) remain valid for the undamaged elongation and bulk modulus o.d.f.s $K_0(\mathbf{N})$ and $\kappa_0(\mathbf{N})$ associated with $\mathbb{K}_0$, on condition that

$$u_0 = \alpha_1 + 2\alpha_2 + \tfrac{1}{5}\alpha_3 + \tfrac{2}{3}\alpha_4 + \tfrac{4}{3}\alpha_5 \tag{79a}$$

$$v_0 = 3\alpha_1 + 2\alpha_2 + \tfrac{1}{3}\alpha_3 + 2\alpha_4 + \tfrac{4}{3}\alpha_5 \tag{79b}$$

$$\mathbf{U}'_0 = -(\tfrac{2}{7}\alpha_3 + \tfrac{2}{3}\alpha_4 + \tfrac{4}{3}\alpha_5)\mathbf{I} + (\tfrac{6}{7}\alpha_3 + 2\alpha_4 + 4\alpha_5)\mathbf{M} \tag{79c}$$

$$\mathbf{V}'_0 = -(\tfrac{1}{3}\alpha_3 + \alpha_4 + \tfrac{4}{3}\alpha_5)\mathbf{I} + (\alpha_3 + 3\alpha_4 + 4\alpha_5)\mathbf{M}, \tag{79d}$$

which are calculated by means of eqns (41c) and (41d).

7.2. *Damage variables and damaged elasticity tensor*

It was experimentally observed that, in a large number of unidirectional fibre-reinforced composites subjected to monotone or cyclic loading, typical damage systems consist of microdefects that are either *parallel* or *perpendicular* to the fibre axis (Talreja, 1985, 1991; He and Sidoroff, 1989). An analysis based on this observation and taking the initial material symmetry—transverse isotropy—into account led us to conclude that, *in the general case, a damaged unidirectional fibre-reinforced composite is monoclinical with respect to the plane normal to the fibre axis* (He and Sidoroff, 1989). In other words, the damaged properties of a unidirectional fibre composite are invariant under any mirror reflection on the plane perpendicular to $\mathbf{m}$.

With the previous symmetry condition in mind, we introduce *two second-order damage tensors* $\mathbf{D}$ and $\mathbf{H}$, which are symmetric positive semi-definite and admit the following spectral decompositions

$$\mathbf{D} = D_1\mathbf{m}\otimes\mathbf{m} + D_2\mathbf{d}_2\otimes\mathbf{d}_2 + D_3\mathbf{d}_3\otimes\mathbf{d}_3, \tag{80a}$$

$$\begin{aligned}\mathbf{H} &= H_1\mathbf{m}\otimes\mathbf{m} + H_2\mathbf{h}_2\otimes\mathbf{h}_2 + H_3\mathbf{h}_3\otimes\mathbf{h}_3\\ &= H_1\mathbf{m}\otimes\mathbf{m} + H_{22}\mathbf{d}_2\otimes\mathbf{d}_2 + H_{33}\mathbf{d}_3\otimes\mathbf{d}_3 + H_{23}\mathbf{d}_2\otimes\mathbf{d}_3 + H_{32}\mathbf{d}_3\otimes\mathbf{d}_2,\end{aligned} \tag{80b}$$

where D_1, D_2 and D_3 are the eigenvalues of $\mathbf{D}$, H_1, H_2 and H_3 are the eigenvalues of $\mathbf{H}$, and both $\{\mathbf{m}, \mathbf{d}_2, \mathbf{d}_3\}$ and $\{\mathbf{m}, \mathbf{h}_2, \mathbf{h}_3\}$ form three-dimensional orthonormal bases. Then, the damage o.d.f.s $\hat{d}(\mathbf{N})$ and $\hat{\eta}(\mathbf{N})$ are assumed to be such that

$$\hat{d}(\mathbf{N}) = \delta + \mathbf{D}' : \mathbf{F}(\mathbf{N}) \tag{81a}$$

$$\hat{\eta}(\mathbf{N}) = h + \mathbf{H}' : \mathbf{F}(\mathbf{N}) \tag{81b}$$

in which

$$\delta = \tfrac{1}{3}\operatorname{tr}\mathbf{D}, \quad h = \tfrac{1}{3}\operatorname{tr}\mathbf{H}, \quad \mathbf{D}' = \mathbf{D} - \delta\mathbf{I}, \quad \mathbf{H}' = \mathbf{H} - h\mathbf{I}. \tag{81c}$$

Geometrically, either $\hat{d}(\mathbf{N})$ or $\hat{\eta}(\mathbf{N})$ describes an ellipsoid with one principal axis parallel to $\mathbf{m}$, when $\mathbf{N}\in\mathcal{N}$ varies and provided the eigenvalues of $\mathbf{D}$ and $\mathbf{H}$ are all positive.

In accordance with eqns (80a) and (80b), the general expressions (45a) and (45b) reduce to

$$\begin{aligned}\bar{K}(\mathbf{N}) = (1-\delta)u_0 + [(1-\delta)\mathbf{U}'_0 - u_0\mathbf{D}'] : \mathbf{F}(\mathbf{N}) + (1-\delta)\mathbb{K}'_0 :: \mathbb{F}(\mathbf{N})\\ - (\mathbf{D}'\otimes\mathbf{U}'_0) :: (\mathbf{F}(\mathbf{N})\otimes\mathbf{F}(\mathbf{N})) - (\mathbb{K}'_0\otimes\mathbf{D}') ::: (\mathbb{F}(\mathbf{N})\otimes\mathbf{F}(\mathbf{N})),\end{aligned} \tag{82a}$$

$$\bar{\kappa}(\mathbf{N}) = (1-h)v_0 + [(1-h)\mathbf{V}'_0 - v_0\mathbf{H}'] : \mathbf{F}(\mathbf{N}) - (\mathbf{H}'\otimes\mathbf{V}'_0) :: (\mathbf{F}(\mathbf{N})\otimes\mathbf{F}(\mathbf{N})). \tag{82b}$$

In the sense of eqn (55), these two functions can be approximated in the following way:

$$\tilde{K}(\mathbf{N}) \approx \tilde{u}+\bar{\mathbf{U}}':\mathbf{F}(\mathbf{N})+\tilde{\mathbb{K}}'::\mathbb{F}(\mathbf{N}) \tag{83a}$$

$$\tilde{\kappa}(\mathbf{N}) \approx \tilde{v}+\tilde{\mathbf{V}}':\mathbf{F}(\mathbf{N}), \tag{83b}$$

where the coefficients are determined by applying formula (48):

$$\tilde{u} = (1-\delta)u_0-\tfrac{2}{15}\mathbf{D}':\mathbf{U}_0', \quad \tilde{v} = (1-h)v_0-\tfrac{2}{15}\mathbf{H}':\mathbf{V}_0' \tag{84a}$$

$$\bar{\mathbf{U}}' = (1-\delta)\mathbf{U}_0'-u_0\mathbf{D}'+\tfrac{4}{21}\mathrm{tr}\,(\mathbf{D}'\mathbf{U}_0')\,\mathbf{I}-\tfrac{2}{7}(\mathbf{D}'\mathbf{U}_0'+\mathbf{U}_0'\mathbf{D}')-\tfrac{4}{21}\mathbb{K}_0'\mathbf{D}' \tag{84b}$$

$$\tilde{\mathbf{V}}' = (1-h)\mathbf{V}_0'-v_0\mathbf{H}'+\tfrac{4}{21}\,\mathrm{tr}\,(\mathbf{H}'\mathbf{V}_0')\,\mathbf{I}-\tfrac{2}{7}(\mathbf{H}'\mathbf{V}_0'+\mathbf{V}_0'\mathbf{H}') \tag{84c}$$

$$\begin{aligned}
\tilde{\mathbb{K}}' = {} & (1-\delta)\mathbb{K}_0'-\tfrac{2}{105}\,\mathrm{tr}\,(\mathbf{D}'\mathbf{U}_0')\,(\mathbf{I}\otimes\mathbf{I}+\mathbf{I}\,\underline{\otimes}\,\mathbf{I}+\mathbf{I}\,\bar{\otimes}\,\mathbf{I}) \\
& -\tfrac{1}{6}(\mathbf{D}'\otimes\mathbf{U}_0'+\mathbf{U}_0'\otimes\mathbf{D}'+\mathbf{D}'\,\underline{\otimes}\,\mathbf{U}_0'+\mathbf{U}_0'\,\underline{\otimes}\,\mathbf{D}'+\mathbf{D}'\,\bar{\otimes}\,\mathbf{U}_0'+\mathbf{U}_0'\,\bar{\otimes}\,\mathbf{D}') \\
& +\tfrac{1}{21}[\mathbf{I}\otimes(\mathbf{D}'\mathbf{U}_0'+\mathbf{U}_0'\mathbf{D}')+(\mathbf{D}'\mathbf{U}_0'+\mathbf{U}_0'\mathbf{D}')\otimes\mathbf{I}+\mathbf{I}\,\underline{\otimes}\,(\mathbf{D}'\mathbf{U}_0'+\mathbf{U}_0'\mathbf{D}') \\
& +(\mathbf{D}'\mathbf{U}_0')+\mathbf{U}_0'\mathbf{D}')\,\underline{\otimes}\,\mathbf{I}+\mathbf{I}\,\bar{\otimes}\,(\mathbf{D}'\mathbf{U}_0'+\mathbf{U}_0'\mathbf{D}')+(\mathbf{D}'\mathbf{U}_0'+\mathbf{U}_0'\mathbf{D}')\,\bar{\otimes}\,\mathbf{I}] \\
& +\tfrac{4}{77}[\mathbf{I}\otimes(\mathbb{K}_0'\mathbf{D}')+(\mathbb{K}_0'\mathbf{D}')\otimes\mathbf{I}+\mathbf{I}\,\underline{\otimes}\,(\mathbb{K}_0'\mathbf{D}')+(\mathbb{K}_0'\mathbf{D}')\,\underline{\otimes}\,\mathbf{I} \\
& +\mathbf{I}\,\bar{\otimes}\,(\mathbb{K}_0'\mathbf{D}')+(\mathbb{K}_0'\mathbf{D}')\,\bar{\otimes}\,\mathbf{I}] \\
& -\tfrac{2}{11}[(\mathbf{I}\,\underline{\otimes}\,\mathbf{D}')\mathbb{K}_0'+\mathbb{K}_0'(\mathbf{I}\,\underline{\otimes}\,\mathbf{D}')+(\mathbf{D}'\,\underline{\otimes}\,\mathbf{I})\mathbb{K}_0'+\mathbb{K}_0'(\mathbf{D}'\,\underline{\otimes}\,\mathbf{I})].
\end{aligned} \tag{84d}$$

The main steps towards getting these expressions are explained in Appendix 2.

If eqns (84a)–(84d) are substituted into eqn (49), we obtain the damaged elasticity tensor $\mathbb{K}$ in its invariant form, which is approximately associated with $\tilde{K}(\mathbf{N})$ and $\tilde{\kappa}(\mathbf{N})$ in the sense of eqn (58). As in the case of initially isotropic materials, it is useful and otherwise more convenient to write $\mathbb{K}$ in matrix form. For doing so, we employ eqns (69) and (70) with $\mathbf{d}_1 = \mathbf{m}$. Then, the damaged stress–strain relation takes the following matrix form:

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} & \tilde{K}_{13} & \tilde{K}_{14} & & \\ \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} & \tilde{K}_{24} & & \\ \tilde{K}_{31} & \tilde{K}_{32} & \tilde{K}_{33} & \tilde{K}_{34} & & \\ \tilde{K}_{41} & \tilde{K}_{42} & \tilde{K}_{43} & \tilde{K}_{44} & & \\ & & & & \tilde{K}_{55} & \tilde{K}_{56} \\ & & & & \tilde{K}_{65} & \tilde{K}_{66} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{bmatrix} \tag{85}$$

with

$$\begin{aligned}
\tilde{K}_{11} &= (1-D_1)(\alpha_1+2\alpha_2+2\alpha_4+4\alpha_5)+(1-\tfrac{215}{231}D_1-\tfrac{8}{231}D_2-\tfrac{8}{231}D_3)\alpha_3 \\
\tilde{K}_{22} &= (1-D_2)(\alpha_1+2\alpha_2)-\tfrac{1}{231}(5D_1-6D_2+D_3)\alpha_3 \\
\tilde{K}_{33} &= (1-D_3)(\alpha_1+2\alpha_2)-\tfrac{1}{231}(5D_1+D_2-6D_3)\alpha_3 \\
\tilde{K}_{44} &= \tfrac{1}{2}(D_1-2D_2-2D_3-3H_1+3H_{22}+3H_{33})\alpha_1 \\
&\quad +(2+D_1-2D_2-2D_3-H_1+H_{22}+H_{33})\alpha_2 \\
&\quad +(\tfrac{65}{154}D_1+\tfrac{6}{154}D_2+\tfrac{6}{154}D_3-\tfrac{33}{70}H_1-\tfrac{1}{70}H_{22}-\tfrac{1}{70}H_{33})\alpha_3 \\
&\quad +(D_1-\tfrac{67}{35}H_1+\tfrac{16}{35}H_{22}+\tfrac{16}{35}H_{33})\alpha_4 \\
&\quad +(2D_1-\tfrac{66}{35}H_1-\tfrac{3}{35}H_{22}-\tfrac{3}{35}H_{33})\alpha_5
\end{aligned}$$

$$\begin{aligned}\tilde{K}_{55} = & -\tfrac{1}{2}(2D_1 - D_2 + 2D_3 - 3H_1 + 3H_{22} - 3H_{33})\alpha_1 \\ & + (2 - 2D_1 + D_2 - 2D_3 + H_1 - H_{22} + H_{33})\alpha_2 \\ & - (\tfrac{40}{154}D_1 + \tfrac{1}{154}D_2 + \tfrac{36}{154}D_3 - \tfrac{27}{70}H_1 + \tfrac{1}{70}H_{22} - \tfrac{9}{70}H_{33})\alpha_3 \\ & - (D_1 + D_3 - \tfrac{58}{35}H_1 + \tfrac{19}{35}H_{22} - \tfrac{31}{35}H_{33})\alpha_4 \\ & + (2 - 2D_1 - 2D_3 + \tfrac{54}{35}H_1 - \tfrac{3}{35}H_{22} + \tfrac{18}{35}H_{33})\alpha_5\end{aligned}$$

$$\begin{aligned}\tilde{K}_{66} = & -\tfrac{1}{2}(2D_1 + 2D_2 - D_3 - 3H_1 - 3H_{22} + 3H_{33})\alpha_1 \\ & + (2 - 2D_1 - 2D_2 + D_3 + H_1 + H_{22} - H_{33})\alpha_2 \\ & - (\tfrac{40}{154}D_1 + \tfrac{36}{154}D_2 + \tfrac{1}{154}D_3 - \tfrac{27}{70}H_1 - \tfrac{9}{70}H_{22} + \tfrac{1}{70}H_{33})\alpha_3 \\ & - (D_1 + D_2 - \tfrac{58}{35}H_1 - \tfrac{31}{35}H_{22} + \tfrac{19}{35}H_{33})\alpha_4 \\ & + (2 - 2D_1 - 2D_2 + \tfrac{54}{35}H_1 + \tfrac{18}{35}H_{22} - \tfrac{3}{35}H_{33})\alpha_5\end{aligned}$$

$$\begin{aligned}\tilde{K}_{12} = & \frac{1}{\sqrt{2}}(2 + D_1 + D_2 - D_3 - 3H_1 - 3H_{22} + 3H_{33})\alpha_1 \\ & + \sqrt{2}(D_1 + D_2 - D_3 - H_1 - H_{22} + H_{33})\alpha_2 \\ & + \frac{1}{\sqrt{2}}(\tfrac{215}{231}D_1 + \tfrac{1}{231}D_2 + \tfrac{15}{231}D_3 - \tfrac{27}{35}H_1 - \tfrac{9}{35}H_{22} + \tfrac{1}{35}H_{33})\alpha_3 \\ & + \sqrt{2}(1 + D_1 - \tfrac{58}{35}H_1 - \tfrac{31}{35}H_{22} + \tfrac{19}{35}H_{33})\alpha_4 \\ & + \sqrt{2}(D_1 - \tfrac{54}{35}H_1 - \tfrac{18}{35}H_{22} + \tfrac{3}{35}H_{33})\alpha_5\end{aligned}$$

$$\begin{aligned}\tilde{K}_{13} = & \frac{1}{\sqrt{2}}(2 + D_1 - D_2 + D_3 - 3H_1 + 3H_{22} - 3H_{33})\alpha_1 \\ & + \sqrt{2}(D_1 - D_2 + D_3 - H_1 + H_{22} - H_{33})\alpha_2 \\ & + \frac{1}{\sqrt{2}}(\tfrac{215}{231}D_1 + \tfrac{15}{231}D_2 + \tfrac{1}{231}D_3 - \tfrac{27}{35}H_1 + \tfrac{1}{35}H_{22} - \tfrac{9}{35}H_{33})\alpha_3 \\ & + \sqrt{2}(1 + D_1 - \tfrac{58}{35}H_1 + \tfrac{19}{35}H_{22} - \tfrac{31}{35}H_{33})\alpha_4 \\ & + \sqrt{2}(D_1 - \tfrac{54}{35}H_1 + \tfrac{3}{35}H_{22} - \tfrac{18}{35}H_{33})\alpha_5\end{aligned}$$

$$\tilde{K}_{14} = -\sqrt{2}H_{23}(3\alpha_1 + 2\alpha_2 + \tfrac{1}{7}\alpha_3 + \tfrac{10}{7}\alpha_4 + \tfrac{4}{7}\alpha_5)$$

$$\begin{aligned}\tilde{K}_{23} = & \frac{1}{\sqrt{2}}(2 - D_1 + D_2 + D_3 + 3H_1 - 3H_{22} - 3H_{33})\alpha_1 \\ & - \sqrt{2}(D_1 - D_2 - D_3 - H_1 + H_{22} + H_{33})\alpha_2 \\ & - \frac{1}{\sqrt{2}}(\tfrac{205}{231}D_1 + \tfrac{13}{231}D_2 + \tfrac{13}{231}D_3 - \tfrac{33}{35}H_1 - \tfrac{1}{35}H_{22} - \tfrac{1}{35}H_{33})\alpha_3 \\ & - \sqrt{2}(D_1 - \tfrac{67}{35}H_1 + \tfrac{16}{35}H_{22} - \tfrac{16}{35}H_{33})\alpha_4 \\ & - \sqrt{2}(2D_1 - \tfrac{66}{35}H_1 - \tfrac{2}{35}H_{22} - \tfrac{2}{35}H_{33})\alpha_5\end{aligned}$$

$$\tilde{K}_{24} = \tilde{K}_{34} = 0$$

$$\tilde{K}_{56} = \sqrt{2}H_{23}(\tfrac{3}{2}\alpha_1 + \alpha_2 + \tfrac{1}{14}\alpha_3 + \tfrac{5}{7}\alpha_4 + \tfrac{2}{7}\alpha_5).$$

In the above expressions, H_1, H_{22}, H_{33} and H_{23} (or H_{32}) denote the non-zero components of **H** relative to the basis $\{\mathbf{m}, \mathbf{d}_2, \mathbf{d}_3\}$, and appearance of $\sqrt{2}$ is due to the use of eqns (69) and (70).

It is seen from eqn (35) that $\tilde{\mathbb{K}}$ is *monoclinic* with respect to the fibre axis specified by **m**. This arises from eqns (80) and (81) and can be deduced already from eqns (82a) and (82b). Indeed, noting that

$$\mathscr{G} := \{\mathbf{Q} \in \mathscr{O}(3) | \mathbf{QMQ}^{\mathrm{T}} = \mathbf{M},\ \mathbf{Qd}_2 = \mathbf{d}_2,\ \mathbf{Qd}_3 = \mathbf{d}_3\} = \{\mathbf{I}, \mathbf{I} - 2\mathbf{M}\} \tag{86}$$

is a *monoclinic symmetry group*, then it is not difficult to verify that, with eqns (80), (82a) and (82b),

$$\bar{K}(\mathbf{QNQ}^{\mathrm{T}}) = \bar{K}(\mathbf{N}), \quad \bar{\kappa}(\mathbf{QNQ}^{\mathrm{T}}) = \bar{\kappa}(\mathbf{N}), \quad \forall \mathbf{Q} \in \mathscr{G}. \tag{87}$$

Nevertheless, as the five undamaged constants $\alpha_1, \ldots, \alpha_5$ are given, $\tilde{\mathbb{K}}$ depends only on seven independent parameters $\{D_1, D_2, D_3;\ H_1, H_{22}, H_{33}, H_{23}\}$ and the monoclinic symmetry in question is not the general one, where $\tilde{\mathbb{K}}$ should depend on 13 independent parameters.

In addition, when $\mathbf{DH} = \mathbf{HD}$ or equivalently when $H_{23} = 0$, $H_{22} = H_2$ and $H_{33} = H_3$ in eqn (85), $\tilde{\mathbb{K}}$ becomes an orthotropic elasticity tensor containing six independent parameters. The resulting model is useful when loading is such that the strain components E_2, E_3 and E_4 are *proportional*. For instance, if $E_3 = E_4 = E_5 = 0$, this situation occurs and the non-zero strain components are all in the plane $\mathbf{m}$–$\mathbf{d}_2$. The stress–strain relation (85) then reduces to the following two-dimensional one:

$$\begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} = \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} & \\ \tilde{K}_{21} & \tilde{K}_{22} & \\ & & \tilde{K}_{66} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_6 \end{bmatrix}$$

$$S_3 = \tilde{K}_{31} E_1 + \tilde{K}_{32} E_2, \quad S_4 = S_5 = 0,$$

which is orthotropic with respect to the axes defined by **m** and $\mathbf{d}_2$.

8. CONCLUDING REMARKS

In the most widely used continuum damage mechanics theories, the damaged elastic response is formulated via the concept of effective stress (strain) and the hypothesis of strain (stress) or elastic energy equivalence. Such an approach presents two major shortcomings:

(i) The choice of damage variables is to a large extent left arbitrary, both regarding their nature (scalar, vector, second-order tensor, etc.) and their number. There is no doubt that the lack of a mathematical support in guiding that choice is the principal source of this arbitrariness.

(ii) The degree of approximation with which the damaged stress–strain relation is formulated is often uncontrolled and so fuzzy. The main reason for this fuzziness is that the terms reflecting the same degree effect of damage to be retained in the expression of the damaged stress–strain relation are not well identified.

In this paper, a different approach has been developed, which remedies the foregoing shortcomings under the condition that the damaged elastic response at a given damage state can be assumed to be linear and hyperelastic.

While the objective of the present work was to develop a more uniform and rigorous approach to *damaged* elastic stress–strain relations, the results obtained in Sections 3 and 4 have wider scope. For example, Theorem 2 is useful for approximating and identifying the elastic properties of anisotropic materials.

In the near future, the authors intend to

- formulate the evolution equations for the damage variables within the framework of generalized standard materials (see e.g. Germain *et al.*, 1983), so as to complete the developed approach;
- confront the models established in Sections 6 and 7 with the available experimental and micromechanical results, examine their limitations and then integrate them into a code of finite elements;
- compare the proposed approach with that based on the theory of tensor function representations;
- study the possibility of extending our approach so as to take the unilateral effect of damage into account.

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APPENDIX 1: IRREDUCIBLE DECOMPOSITION OF THE TRANSVERSELY ISOTROPIC ELASTICITY TENSOR

We here employ a procedure suggested by Cowin (1989) for making irreducible decomposition of the elasticity $\mathbb{K}_0$ in eqn (76).

Two symmetric second-order tensors $\mathbf{Y}_0$ and $\mathbf{Z}_0$, defined as

$$\mathbf{Y}_{0ij} := \mathbb{K}_{0ijmm}, \quad \mathbf{Z}_{0ij} := \mathbb{K}_{0imjm},$$

can directly be calculated from $\mathbb{K}_0$. After introducing the invariant expression (76) of the transversely isotropic elasticity tensor $\mathbb{K}_0$ into the above definitions, a simple calculation gives

$$\mathbf{Y}_0 = (3\alpha_1+2\alpha_2+\alpha_4)\mathbf{I}+(\alpha_3+3\alpha_4+4\alpha_5)\mathbf{M}$$
$$\mathbf{Z}_0 = (\alpha_1+4\alpha_2+\alpha_5)\mathbf{I}+(\alpha_3+2\alpha_4+5\alpha_5)\mathbf{M}.$$

The corresponding deviatoric tensors are

$$\mathbf{Y}'_0 = -\tfrac{1}{3}(\alpha_3+3\alpha_4+4\alpha_5)\mathbf{I}+(\alpha_3+3\alpha_4+4\alpha_5)\mathbf{M}$$
$$\mathbf{Z}'_0 = -\tfrac{1}{3}(\alpha_3+2\alpha_4+5\alpha_5)\mathbf{I}+(\alpha_3+2\alpha_4+5\alpha_5)\mathbf{M}.$$

Then, using formulae (4.3) and (4.4) of Cowin [1989], we have

$$\alpha_0 = \tfrac{1}{15}(2\,\mathrm{tr}\,\mathbf{Y}_0-\mathrm{tr}\,\mathbf{Z}_0) = \tfrac{1}{15}(15\alpha_1+\alpha_3+10\alpha_4)$$
$$\beta_0 = \tfrac{1}{30}(3\,\mathrm{tr}\,\mathbf{Z}_0-\mathrm{tr}\,\mathbf{Y}_0) = \tfrac{1}{30}(30\alpha_2+2\alpha_3+20\alpha_5)$$
$$\mathbf{A}'_0 = \tfrac{1}{7}(5\mathbf{Y}'_0-4\mathbf{Z}'_0) = -\tfrac{1}{21}(\alpha_3+7\alpha_4)\mathbf{I}+\tfrac{1}{7}(\alpha_3+7\alpha_4)\mathbf{M}$$
$$\mathbf{B}'_0 = \tfrac{1}{7}(3\mathbf{Z}'_0-2\mathbf{Y}'_0) = -\tfrac{1}{21}(\alpha_3+7\alpha_5)\mathbf{I}+\tfrac{1}{7}(\alpha_3+7\alpha_5)\mathbf{M}$$
$$\mathbb{K}'_0 = \mathbb{K}_0-[\alpha_0\mathbf{I}\otimes\mathbf{I}+\beta_0(\mathbf{I}\,\underline{\otimes}\,\mathbf{I}+\mathbf{I}\,\bar{\otimes}\,\mathbf{I})+\mathbf{I}\otimes\mathbf{A}'_0+\mathbf{A}'_0\otimes\mathbf{I}+\mathbf{I}\,\underline{\otimes}\,\mathbf{B}'_0+\mathbf{B}'_0\,\underline{\otimes}\,\mathbf{I}+\mathbf{I}\,\bar{\otimes}\,\mathbf{B}'_0+\mathbf{B}'_0\,\bar{\otimes}\,\mathbf{I}]$$
$$= \alpha_3[\mathbf{M}\otimes\mathbf{M}-\tfrac{1}{7}(\mathbf{I}\otimes\mathbf{M}+\mathbf{M}\otimes\mathbf{I}+\mathbf{I}\,\underline{\otimes}\,\mathbf{M}+\mathbf{M}\,\underline{\otimes}\,\mathbf{I}+\mathbf{I}\,\bar{\otimes}\,\mathbf{M}+\mathbf{M}\,\bar{\otimes}\,\mathbf{I})+\tfrac{1}{35}(\mathbf{I}\otimes\mathbf{I}+\mathbf{I}\,\underline{\otimes}\,\mathbf{I}+\mathbf{I}\,\bar{\otimes}\,\mathbf{I})].$$

APPENDIX 2: CALCULATION OF THE EXPANSION COEFFICIENTS ASSOCIATED WITH THE FUNDAMENTAL SPHERICAL HARMONICS

The expressions of $\bar{K}(\mathbf{N})$ and $\bar{\kappa}(\mathbf{N})$ are given by eqns (82a) and (82b). In the following, we present the key formulae and steps needed for carrying out the integrals of eqn (48) and thus obtaining the expansion coefficients $\{\bar{u},\bar{\mathbf{U}}',\bar{\mathbb{K}}'\}$ and $\{\bar{v},\bar{\mathbf{V}}'\}$ of eqns (83a) and (83b).

First, recall the following recurrence formulae for integrals over the unit sphere $\mathscr{S}$ (see e.g. Kanatani, 1984):

$$\frac{1}{4\pi}\int_{\mathscr{S}} \mathrm{d}a = 1, \quad \frac{1}{4\pi}\int_{\mathscr{S}} n_i n_j\,\mathrm{d}a = \tfrac{1}{3}\delta_{ij}, \quad \frac{1}{4\pi}\int_{\mathscr{S}} n_i n_j n_k n_l\,\mathrm{d}a = \tfrac{1}{5}J_{ijkl}$$
$$\frac{1}{4\pi}\int_{\mathscr{S}} n_i n_j n_k n_l n_m n_n\,\mathrm{d}a = \tfrac{1}{7}J_{ijklmn}, \quad \frac{1}{4\pi}\int_{\mathscr{S}} n_i n_j n_k n_l n_m n_p n_q\,\mathrm{d}a = \tfrac{1}{9}J_{ijklmnpq}$$
$$\frac{1}{4\pi}\int_{\mathscr{S}} n_i n_j n_k n_l n_m n_n n_p n_q n_r n_s\,\mathrm{d}a = \tfrac{1}{11}J_{ijklmnpqrs},$$

where δ_{ij} is the Kronecker delta and

$$J_{ijkl} = \tfrac{1}{3}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$
$$J_{ijklmn} = \tfrac{1}{5}(\delta_{ij}J_{klmn}+\delta_{ik}J_{jlmn}+\delta_{il}J_{jkmn}+\delta_{im}J_{jkln}++\delta_{in}J_{jklm})$$
$$J_{ijklmnpq} = \tfrac{1}{7}(\delta_{ij}J_{klmnpq}+\delta_{ik}J_{jlmnpq}+\delta_{il}J_{jkmnpq}+\delta_{im}J_{jklnpq}+\delta_{in}J_{jklmpq}+\delta_{ip}J_{jklmnq}+\delta_{iq}J_{jklmnp})$$
$$J_{ijklmnpqrs} = \tfrac{1}{9}(\delta_{ij}J_{klmnpqrs}+\delta_{ik}J_{jlmnpqrs}+\delta_{il}J_{jkmnpqrs}+\delta_{im}J_{jklnpqrs}$$
$$+\delta_{in}J_{jklmpqrs}+\delta_{ip}J_{jklmnqrs}+\delta_{ir}J_{jklmnprs}+\delta_{in}J_{jklmnpqs}+\delta_{is}J_{jklmnpqr}).$$

Substituting eqns (82a) and (82b) into eqn (48), while taking account of the orthogonality relations of eqn (9), we get

$$\bar{u} = \frac{1}{4\pi}\int_{\mathscr{S}} \bar{K}(\mathbf{N})\,\mathrm{d}a = (1-\delta)u_0-\left[\frac{1}{4\pi}\int_{\mathscr{S}} \mathbf{F}(\mathbf{N})\otimes\mathbf{F}(\mathbf{N})\,\mathrm{d}a\right]::(\mathbf{D}'\otimes\mathbf{U}'_0)$$
$$\bar{v} = \frac{1}{4\pi}\int_{\mathscr{S}} \bar{\kappa}(\mathbf{N})\,\mathrm{d}a = (1-h)v_0-\left[\frac{1}{4\pi}\int_{\mathscr{S}} \mathbf{F}(\mathbf{N})\otimes\mathbf{F}(\mathbf{N})\,\mathrm{d}a\right]::(\mathbf{H}'\otimes\mathbf{V}'_0)$$
$$\bar{\mathbf{U}}' = \frac{15}{8\pi}\int_{\mathscr{S}} \bar{K}(\mathbf{N})\mathbf{F}(\mathbf{N})\,\mathrm{d}a$$

$$= (1-\delta)\mathbf{U}'_0 - u_0\mathbf{D}' - \left[\frac{15}{8\pi}\int_{\mathscr{S}} \mathbf{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N})\,\mathrm{d}a\right] :: (\mathbf{D}' \otimes \mathbf{U}'_0)$$

$$- \left[\frac{15}{8\pi}\int_{\mathscr{S}} \mathbf{F}(\mathbf{N}) \otimes \mathbb{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N})\,\mathrm{d}a\right] ::: (\mathbb{K}'_0 \otimes \mathbf{D}')$$

$$\tilde{\mathbf{V}}' = \frac{15}{8\pi}\int_{\mathscr{S}} \tilde{\kappa}(\mathbf{N})\mathbf{F}(\mathbf{N})\,\mathrm{d}a$$

$$= (1-h)\mathbf{V}'_0 - v_0\mathbf{H}' - \left[\frac{15}{8\pi}\int_{\mathscr{S}} \mathbf{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N})\,\mathrm{d}a\right] :: (\mathbf{H}' \otimes \mathbf{V}'_0)$$

$$\tilde{\mathbb{K}}' = \frac{315}{32\pi}\int_{\mathscr{S}} \tilde{K}(\mathbf{N})\mathbb{F}(\mathbf{N})\,\mathrm{d}a$$

$$= (1-\delta)\mathbb{K}'_0 - \left[\frac{315}{32\pi}\int_{\mathscr{S}} \mathbb{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N})\,\mathrm{d}a\right] :: (\mathbf{D}' \otimes \mathbf{U}'_0)$$

$$- \left[\frac{315}{32\pi}\int_{\mathscr{S}} \mathbb{F}(\mathbf{N}) \otimes \mathbb{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N})\,\mathrm{d}a\right] ::: (\mathbb{K}'_0 \otimes \mathbf{D}').$$

Before applying the recurrence formulae to perform the integrals in these expressions, it is useful to note that, since $\mathbf{U}'_0$, $\mathbf{V}'_0$, $\mathbf{D}'$ and $\mathbf{H}'$ are s.t. and $\mathbb{K}'_0$ is c.s.t.,

$$[\mathbf{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N})] :: (\mathbf{D}' \otimes \mathbf{U}'_0) = (\mathbf{N} \otimes \mathbf{N}) :: (\mathbf{D}' \otimes \mathbf{U}'_0)$$

$$[\mathbf{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N})] :: (\mathbf{H}' \otimes \mathbf{V}'_0) = (\mathbf{N} \otimes \mathbf{N}) :: (\mathbf{H}' \otimes \mathbf{V}'_0)$$

$$[\mathbb{F}(\mathbf{N}) \otimes \mathbf{F}(\mathbf{N})] ::: (\mathbb{K}'_0 \otimes \mathbf{D}') = (\mathbf{N} \otimes \mathbf{N} \otimes \mathbf{N}) ::: (\mathbb{K}'_0 \otimes \mathbf{D}').$$

Then, a direct but rather laborious calculation leads to the results (84a)–(84d).